From centralized to decentralized power systems:

Some stochastic optimization problems in energy management.

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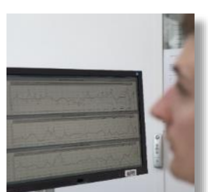
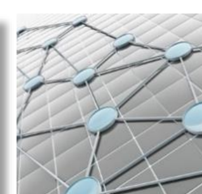
Winter School on Stochastic Programming in Energy

Geilo, February 2018



EDF : A WORLD LEADER IN ENERGY IN A CHANGING WORLD

38 millions
customers in the world

170,000
employees in the world

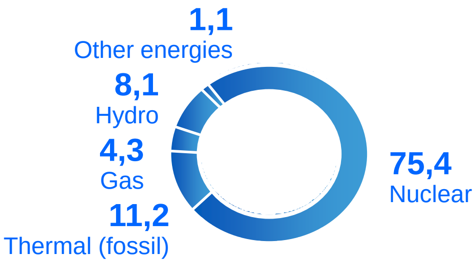
66.3 Billions €
Turnover
(49 % out of France)

117,1 g
of CO_2 per kWh produced

618,5 TWh
produced in the world

- Electricity: generation, transmission, distribution, trading
- Natural Gas

Mix of production 2009 (%)



Changing world :

- Big cities : 50 % of population live in towns, 70 % in 2050
- Emerging Countries : China, Brazil, India…
- Finite ressources and need to « de-carbon » energy
- New ways to generate, to manage, to use electricity→ Local setting with raise of :
  - new and numerous producers (regions, towns, industries, individuals) operating decentralized means of renewable production (wind/solar)
  - new uses of electricity : electrical vehicles, green transportation, data centers, …
  - new management techniques for an optimal use of electricity (demand response, …) based on technologies of information and communication (smart grids) and on mathematics

1/3 of energetical needs are covered by electricity

40 % of electricity in the world comes from **coal**, 20 % from natural **gas**, 16 % from hydro,15 % from nuclear, 7 % from oil and 2 % from renewable energy

MAIN OBJECTIVES OF THE TALK

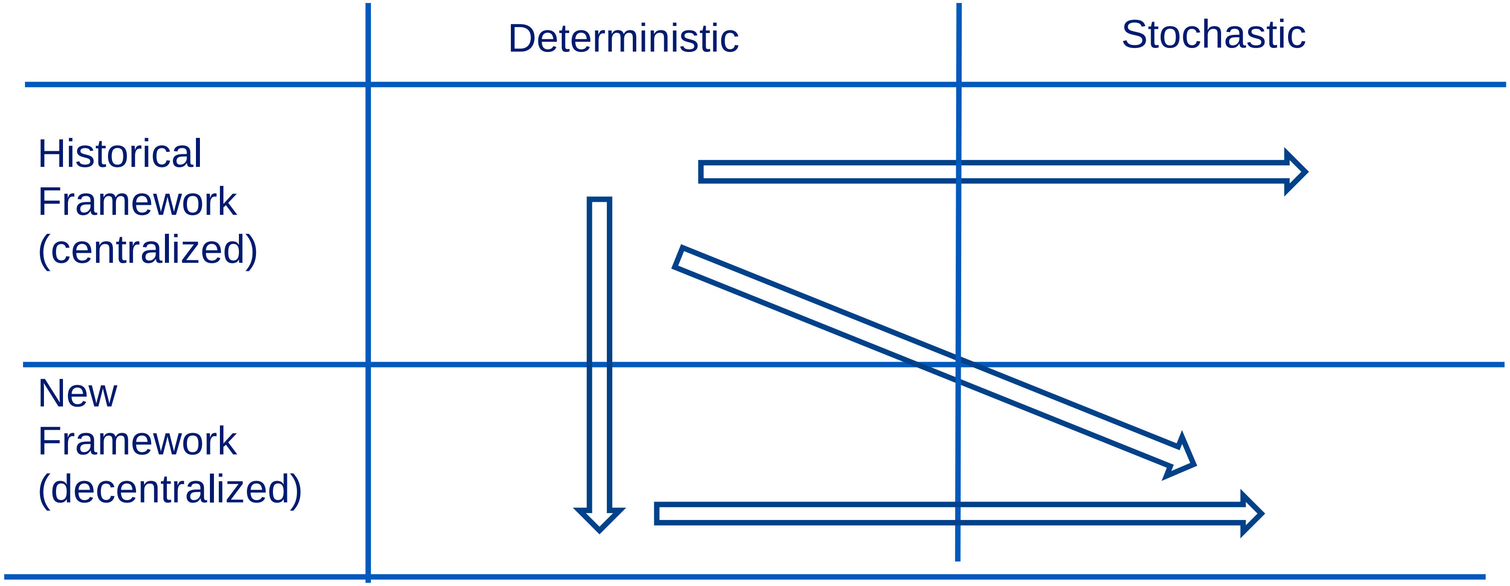
1) To provide a brief introduction to energy management and its evolution

2) To present (not to solve) some stochastic optimization problems in energy management

   a) In the historical centralized framework
   b) In the decentralized framework induced by the evolution of electrical systems

3) To highlight the needs in stochastic optimization from the utility point of view

4) To open the discussion on the different paths for developments in this field



OUTLINE

The framework :

electrical systems and energy management

AN ELECTRICAL SYSTEM

A set of three components electrically connected to each other on a given territory

Generation (the offer, the supply)

Electricity generated by power plants with different characteristics



Load (the demand)

Electricity consumed by

- Industrial/transport customers (23 – 138 KV)
- Commercial customers (4.16 – 34.5 KV)
- Residential customers (120 – 240 V)



Link between supply and demand

Transmission / distribution networks (110-765 KV)

- Transmission network (in France: from 63 kV to 400 kV)
- Distribution network (in France: 230 V to 50 kV)

Frequency

- electrical expression of the rotating speed of the alternator (common parameter to all the components of a system : 50-60 Hz)
- indicator of the gap between supply and demand :
- If demand is higher than supply, the frequency decreases.
- if supply is higher than demand, the frequency increases

Voltage

local parameter of the electrical network: on a given point of the network, the voltage is a function of

- injection (production) : voltage increases
- withdrawal (consumption) : voltages decreases

1. THE LOAD : USES AND USERS OF ELECTRICITY

Electricity consumption depends on customers profiles (demography, economic activity, sectors), networks losses, sale contracts

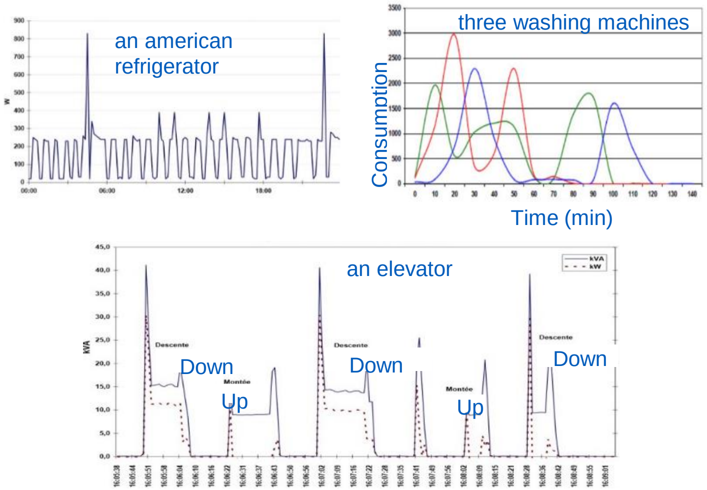
Specific uses (only by electricity).

- freezer, dryer, dishwasher, microwave
- audiovisual TV, DVD player, computer
- Lighting

Competitive uses (substitutions possible). The electricity demand depends on the market shares

- Mainly thermal : heating, domestic hot water, cooking
- Transportation : train vs coach / tram vs bus / electric vs gasoline vehicle

Residential : Some power profiles



EXCHANGES

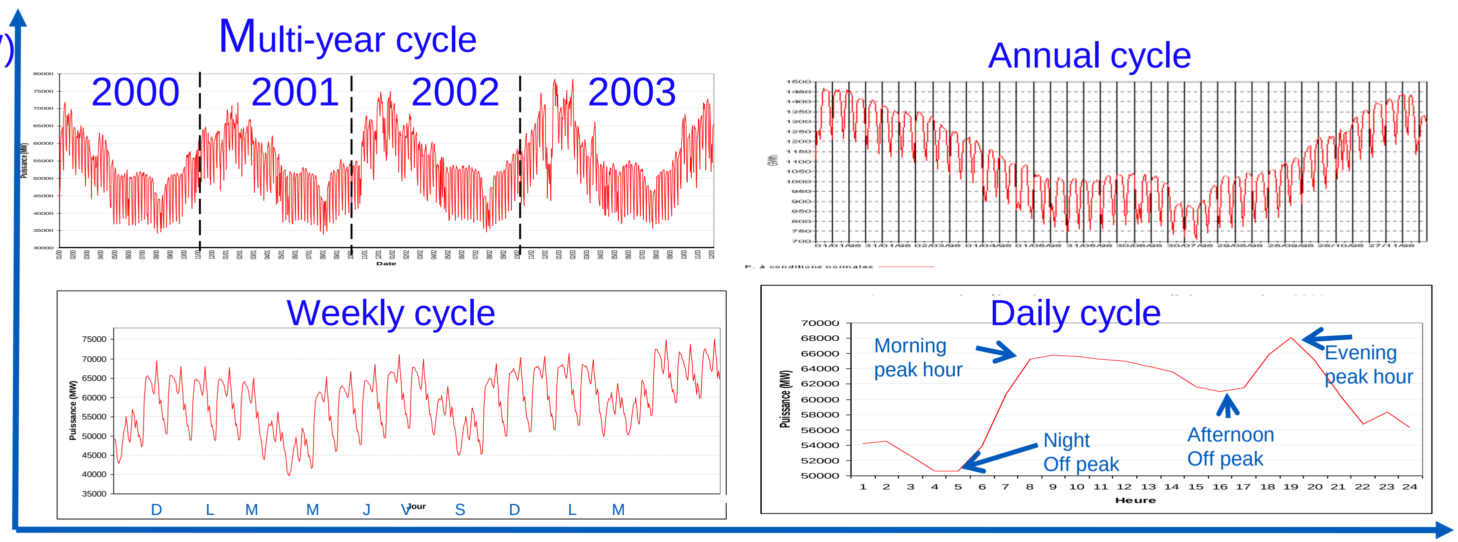
Contracts

LOSSES

Networks

Demand (MW)

Time series with cycles and yearly trend



Time

2. THE SUPPLY : THREE CHARACTERISTICS

Capacity Factor % of production hours expected per year on average over the life of the installation	**Min**	**Max**
Coal	0.5	0.85
IGCC	0.5	0.85
Combustion Turbine	0.88	0.92
Combined Cycle	0.8	0.87
Nuclear	0.70	0.9
Biomass	0.8	0.85
Geothermal (hydrothermal)	0.87	0.95
Wind (onshore)	0.15	0.44
Wind (offshore)	0.30	0.45
Solar Thermal	0.22	0.42
PV	0.10	0.26
Hydro (lake)	0.30	0.50
Hydro (fil de l’eau)	0.80	1.00

Total emissions of CO2		
Nuclear		10 gCO_2/kWh
Renewables	Hydro	6 gCO_2/kWh
	Wind	7 gCO_2/kWh
	Photovoltaic	55 gCO_2/kWh
Fossil fuel based	Coal	1038 gCO_2/kWh
	Fuel	704 gCO_2/kWh
	Gas	406 gCO_2/kWh
	Cogeneration	400 gCO_2/kWh

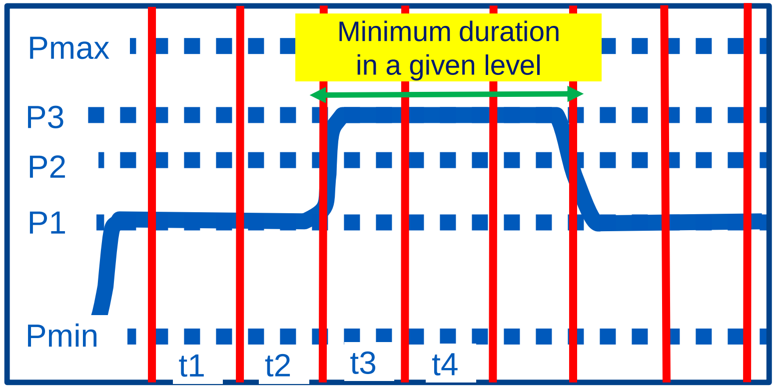
Electricity is very difficult to store

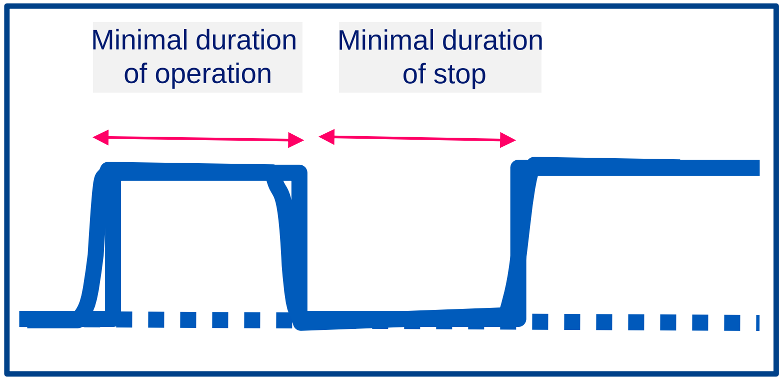
Storage : possible for small amounts of electricity and at very high costs. Three techniques :

- Hydro storage : pumping power station
- Economical incentive «storage» : shedding contracts
- Battery

2. THE SUPPLY : OPERATING A PLANT

- **Starts the plant**
  - start curves: vary depending on the previous stop time
  - Minimal duration of operation
  - Maximum number of start-up per day
- **Produces**
  - Levels of production : Pmin, P1, P2, …, Pmax
  - Minimum duration in a given level of production
  - Maximum number of modulations per day
  - Increases the production (Gradient of power variation upward)
  - Decreases the production (Gradient of power variation downward)
- **Stops the plant**
  - stop curves: vary depending on the previous stop time
  - Minimal duration of stop





4. UNCERTAINTIES ON ELECTRICAL SYSTEMS

- Load curve ← Loss of customers (competition), Exchanges, Economic situation, Customers behavior
- Networks ← Fall of a pylon
- Thermal power plants ← Random outages, Shutdown duration, Temperature of rivers (cooling)
- Hydro systems ← Hydro inflows, Volumes in reservoirs
- Wind farms ← Speed wind, sunshine
- Solar farms ← sunshine, State of the atmosphere

geographic location

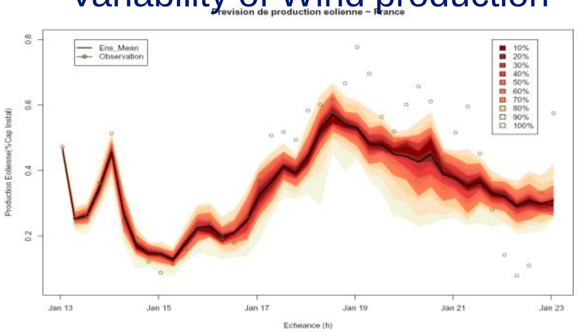
season

Weather Climat

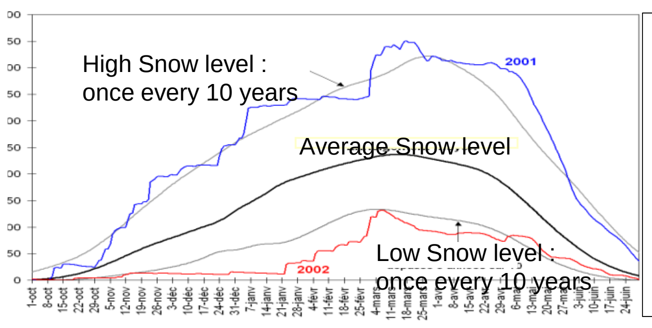
Extreme Phenomenons (drought, flood, …)

Influence/Correlation

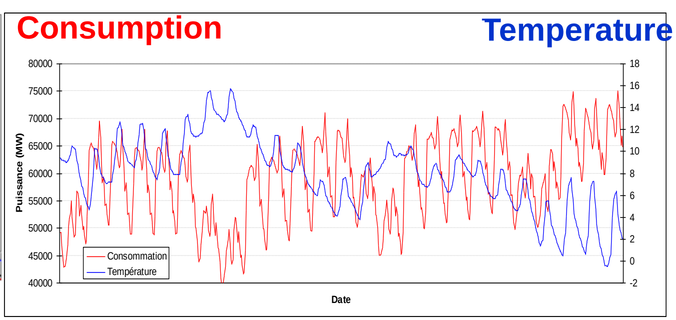
Variability of Wind production



Variability of Hydro Inflows



Load (In winter : -1°C → + 1500 MW)



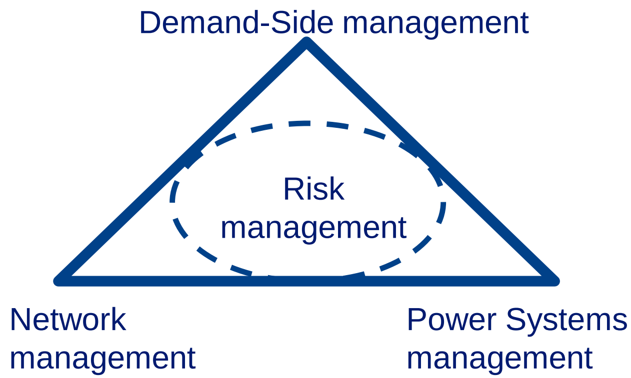
5. ENERGY MANAGEMENT : AT A GLANCE

Energy Management = optimal management of the three components of the electrical system
objective: Efficiency & Cost effectiveness

DECISION PROCESS

- Organization of teams / Timing of decisions
- Decision-making tools & decision-makers

ENERGY MANAGEMENT



Ensuring optimal energy transfert between generation areas and consumption areas at minimal cost and high level of security

Ensuring the balance between supply and demand at minimal cost and at controlled risk.

HISTORICAL LEGAL ISSUE

Monopoly - Integrated Utility

•Generation

•Transmission

•Distribution

PRESENT LEGAL ISSUE

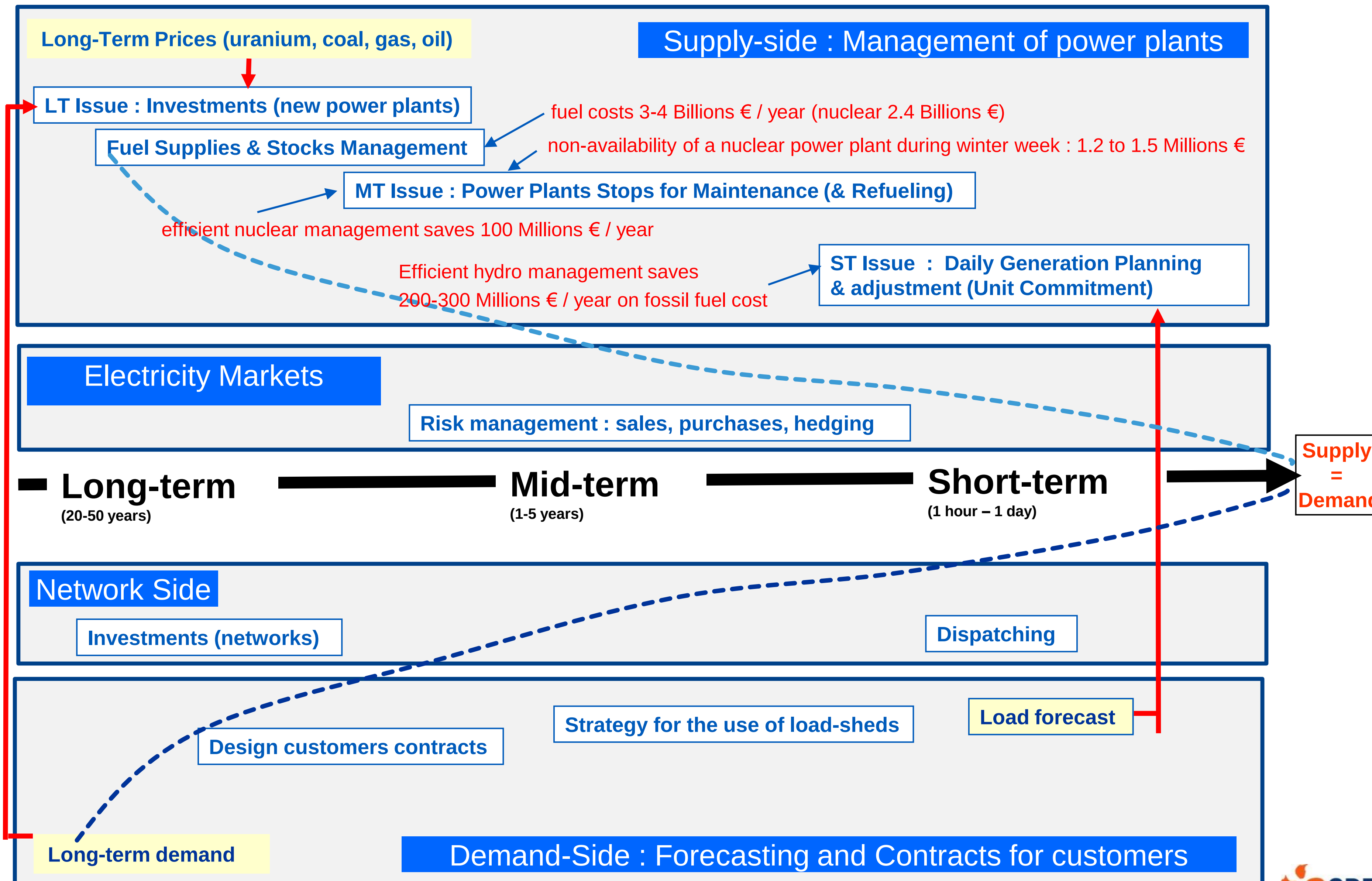
Deregulation, markets

- Generation Companies
- Transmission company
- Distribution companies
- ISO (system reliability)
- Electricity markets
- Prosumers

Management = Software (physical management is excluded)

5. ENERGY MANAGEMENT : OPTIMIZE DECISIONS

Find the best decisions in view to operate the system at short/mid/long termes and at minimal cost



5. ENERGY MANAGEMENT : OPTIMIZATION PROBLEM

A 2-step process

Is the demand too high regarding capacity of production ?

No

Yes

PB 0 : Assessment on network losses, exchanges with others utilities + Decision to start shedding contracts (main tool of flexibility on demand): managed upstream of the optimization process and independently → Demand is a data of an optimization problem

PB 1 : Optimization of production for this demand

Goal : Minimize the global cost of production
Constraints I : Satisfy the demand
Constraints II : Respect the
- technical contraints of power plants
- technical constraints related to the network

MT ← **Statistic** — **Representing uncertainty** — **Laws, Uncertainty sets Scenarios** — ST

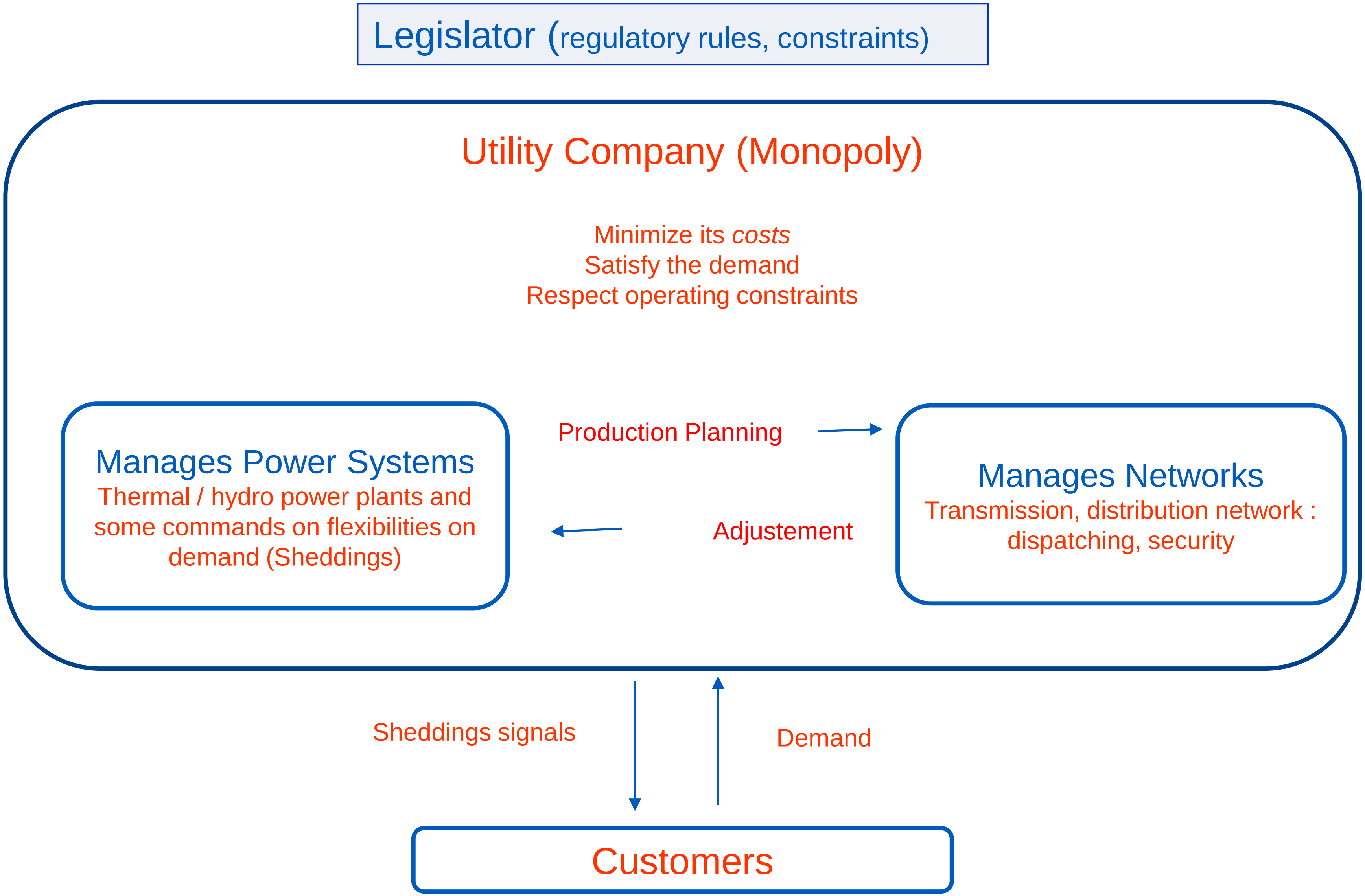
MT — **Simplified** — **Modeling constraints on production units** — **Detailed** → CT

Mixture of 3 challenging features : Uncertainty + Big size + Mixed variables

Some stochastic optimization problems In energy management

Centralized setting

THE CENTRALIZED SETTING



THE UNIT COMMITMENT PROBLEM

The historical short-term problem

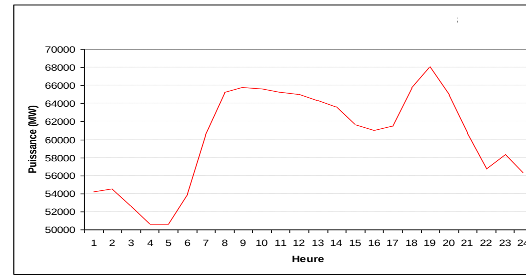
The data of the problem

You have a set of power plants in order to satisfy a given demand on the next 24 hours.

Each power plant is characterized by :

- Costs : generation, starting, …
- Operation constraints

Forecast of the demand on the next 24h



The problem to solve

ST1- Forecast at D-1 the demand for the next 24h of day D

ST2 - Determine at D-1 the best generation planning for the next 24h of day D:

- minimal global cost
- satisfaction of demand
- feasible planning : respects all the operation constraints

ST3 - Adjust at each half an hour of D-Day the initial generation schedule estimated at D-1 with the specific constraint : changes in the planning must only concern a limited number of power plants (typically 30 over 250 power plants)

A generation planning

Power plants →

Time ↓

$$X = \begin{bmatrix} x_{11} & & x_{1n} \\ x_{i1} & x_{ij} & x_{in} \\ x_{m1} & & x_{mn} \end{bmatrix}$$

THE UNIT COMMITMENT PROBLEM

A 2-step process

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→ Demand is a data of the optimization problem

PB 1 : Optimization of production for this demand

Goal : Minimize the global cost of production
Constraints I : Satisfy the demand
Constraints II : Respect the
- technical contraints of production units (Pmin/Pmax / Starts/stops / Minimum Up/Down time / Gradients, …)

Representing uncertainty : integrated in reserves

Modeling constraints on power plants : at short term, a precise description of operation constraints is needed

Mixture of 2 challenging features : Big size + Mixed variables

THE UNIT COMMITMENT PROBLEM : RESOLUTION SCHEME

Detailled description of constraints on power plants – Deterministic setting

$$\min_u \sum_{l\in L}\left(\sum_{t\in T} C_{l,t}\left(u_{l,t},x_{l,t}\right)\right)$$

$$\begin{cases} \forall t \in T, \sum_{l\in L} P_{l,t}\left(u_{l,t},x_{l,t}\right) = d_t,\ \forall t\in T, \boxed{\sum_{l\in L} R_{l,t}\left(u_{l,t},x_{l,t}\right) \ge r_t} \\ \forall l \in L, \left(u_{l,t},x_{l,t}\right) \in D_l \end{cases}$$

Nonconvex, nonlinear, mixed problem of big size

Coordination : bundle method $\sum_{j=1}^{n} p_{ij} = d_i$

Price ⇓⇑ **Production**

59 nuclear power plants
50 thermal plants (coal, fuel, gas)

Price ⇓⇑ **Production**

50 hydro valley

Simple Lagrangian
(compute : marginal costs)

Augmented Lagrangian
(compute : production plannings)

Simplified formulation	Linear Programme (LP)	Quadratic Programme (QP)
Exact Formulation	Mixed Linear Programme (MLP)	Mixed Quadratic Programme (MQP)

Big valley (La Durance) : 13000 constraints, 18000 variables (5000 binaries)
Medium valley (l'Arc) : 12000 constraints, 15000 variables (4000 binaries)
Small valley (la Romanche) : 10000 constraintes, 13000 variables (3000 binaries)

HOW THE ELECTRICAL SYSTEM CAN FACE UNCERTAINTIES ?

The Primary Reserve (Frequency-Power)

- Automated system
- 3000 MW at european level (simultaneous loss of 2 biggest power plants 2x1500 MW)
- Reaction time : few seconds
- In France: from 600 to 800 MW
- 50 % available in 15 s, 100 % in 30 s

The Secondary Reserve (Frequency-Power)

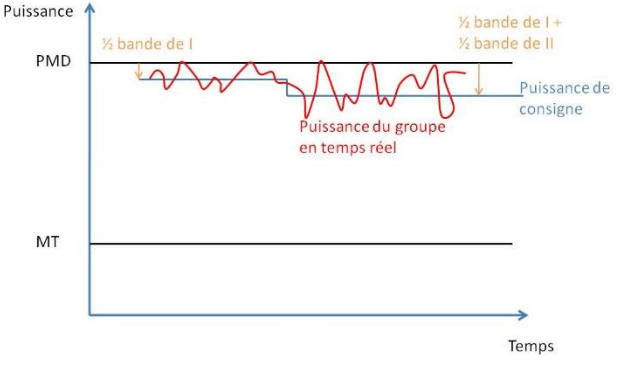
- Automated system
- In France: de 500 à 1000 MW
- Released from 100 to 200 s
- End of mobilization < 15 mn

The Tertiary Reserve

- Human decision
- Fast reserve (hydro) <13 mn : 1000 MW
- Additional reserve (hydro, turbine) <30 mn) : 500 MW

Thermal plants do not operate at maximal capacity !

The setpoint power of a thermal group is between its technical minimum (TM) and its maximum available power (MAP)



THE UCP : COMPUTING THE RESERVES

Probabilistic Approach

$$IP\left[\sum_{j=1}^{N} a_{ij}(\xi_\theta)\, x_{ij} \geq d_i(\xi_\delta)\right] \geq 1-p, \quad i=1,\dots,m$$

a_{ij} Availability coefficient

d Demand

Gaussian setting :

$$\boxed{\sum_{j=1}^{N} IE\left[a_{ij} x_{ij}\right]} + \boxed{\Phi^{-1}(1-p)\sqrt{\sum_{j=1}^{N} \sigma_{ij}^2 x_{ij}^2 + \sigma_{d_i}^2}} \geq IE\left[d_{ij}\right]$$

Mean production

$R_{1\%}$ Reserve for the risk 1%

Hoeffding setting (no hypothesis on laws) :

Lemma 1 : *Any Individual Chance Constraint*

$$\mathbb{P}[\langle A_i(\xi), x\rangle \geq b_i(\xi_\delta)] \geq \alpha_i, \ \forall i \in I$$

is approximated by the 2 (convex) conic quadratic inequalities :

$$\boxed{\langle \mathbb{E}[A_i(\xi)], x\rangle} - \boxed{\sqrt{(1/2)|\ln(1-\alpha_i)|}\, \|\Delta_i x + \delta_{bi}\|_2} \geq \mathbb{E}(b_i)$$

$$\langle \mathbb{E}[A_i(\xi)], x\rangle \geq \mathbb{E}(b_i)$$

$$\delta_{b_i} = (b_i^{max} - b_i^{min})$$

$$\Delta_j = \begin{pmatrix} a_{1j}^{max} - a_{1j}^{min} & \dots & 0 \\ \vdots & a_{ij}^{max} - a_{ij}^{min} & \vdots \\ 0 & \dots & a_{mj}^{max} - a_{mj}^{min} \end{pmatrix}$$

DEALING WITH UNCERTAINTY

Which methods for energy management ?

Simple Scenarios		Uncertainty set	Probability Laws		Tree
↓	. . .	↓	↓	. . .	↓
Easy to implement		**Robust Optimization**	**Chance Constraint Programming**		Stochastic optimization

Investigations : probabilistic models

$$\begin{aligned} \min \quad & f[c(x,\xi)] \\ s.t. \quad & \mathbb{P}[A(\xi)x \geq b(\xi)] \geq 1-p \\ & Px \leq h \\ & x \in X \end{aligned}$$

assets $j = 1, \dots, n$

$$A(\xi) = \begin{pmatrix} A^{\theta}(\xi_{\theta}) & A^{\eta}(\xi_{\eta}) & A^{\mu}(\xi_{\mu}) & A^{\sigma}(\xi_{\sigma}) & A^{\epsilon}(\xi_{\epsilon}) \end{pmatrix}$$

time $i = 1, \dots, m$

$$A^{\alpha}(\xi_{\alpha}) = \begin{pmatrix} a^{\alpha}_{11} & \dots & a^{\alpha}_{1N^{\alpha}} & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ & & & \ddots & & & & \ddots & & & \\ 0 & \dots & 0 & \dots & a^{\alpha}_{i1} & \dots & a^{\alpha}_{iN^{\alpha}} & \dots & 0 & \dots & 0 \\ & & & \ddots & & & & \ddots & & & \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & a^{\alpha}_{m1} & \dots & a^{\alpha}_{mN^{\alpha}} \end{pmatrix}$$

$$\max_{x \in X} IP\left[a \leq Ax \leq b\right]$$

Applications : integration of renewables, statistics, portfolio, inverse problems, …

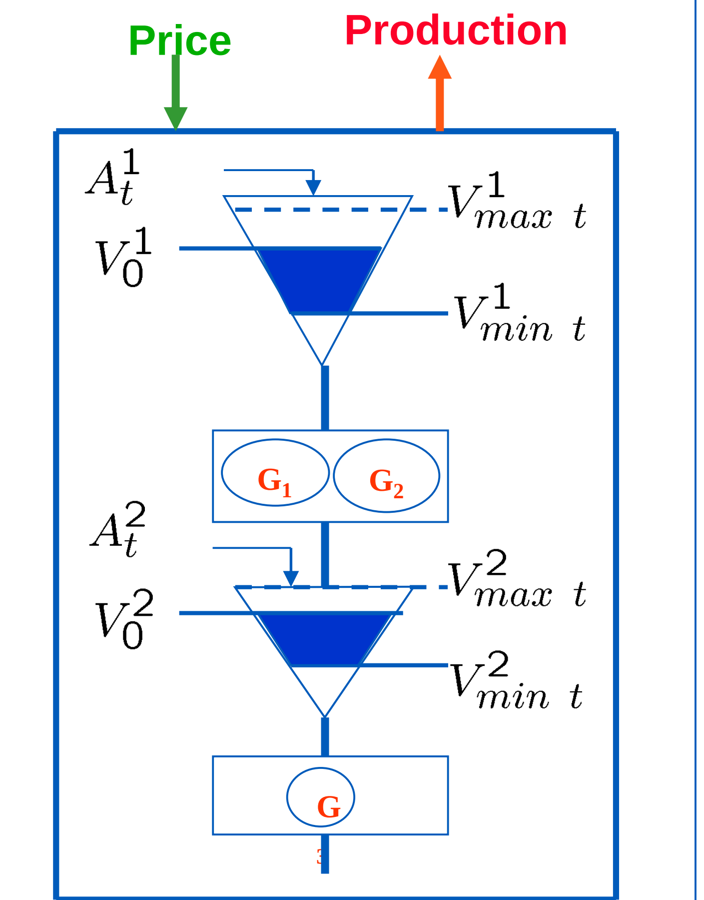
EXPERIMENT - THE HYDRO SUB-PROBLEM OF THE UCP

To determine the production of a valley associated to a price signal

- Cost function : water values (pre-computed)

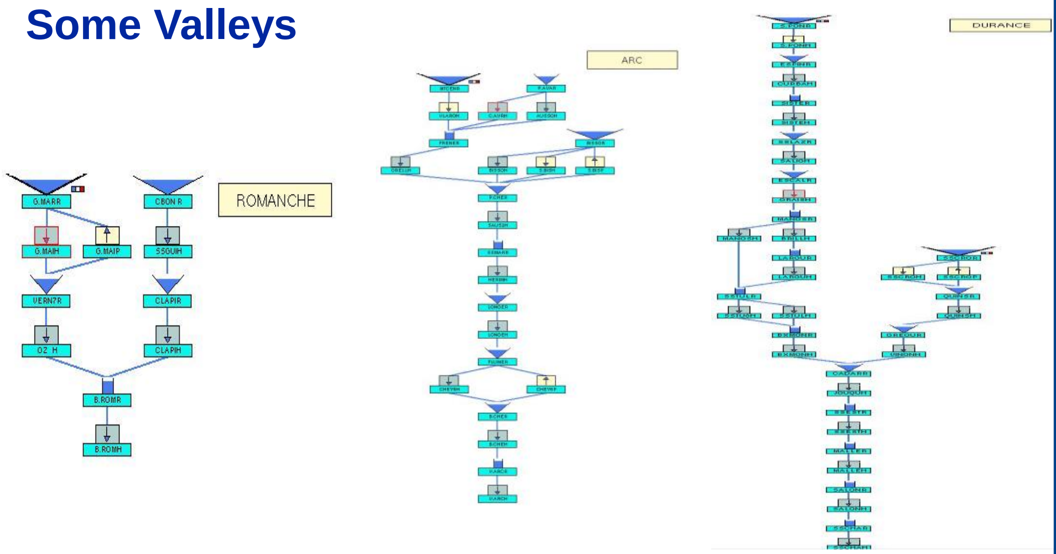
- Constraints (simplified situation):
  - Flow constraints
  - Production level bounds
  - Reservoir bounds (exploitation policy, security)

- Random Inflows: we assume that they follow some causal time series model with Gaussian innovations (correlated between reservoirs)



$$v_t = v_{t-1} - x_t + A_t$$

Some Valleys



EDF

EXPERIMENT - THE HYDRO SUB-PROBLEM OF THE UCP

Probabilistic/Robust Approaches

F1 : Deterministic

F2 : Joint Chance Constraint

$$
\begin{array}{ll}
\min_{x \geq 0} & c^{\top} x \\
s.t. & Ax \leq b \\
 & p \leq \mathbb{P}[a^r + A^r x \leq \eta \leq b^r + B^r x].
\end{array}
$$

F3 : Individual Chance Constraints

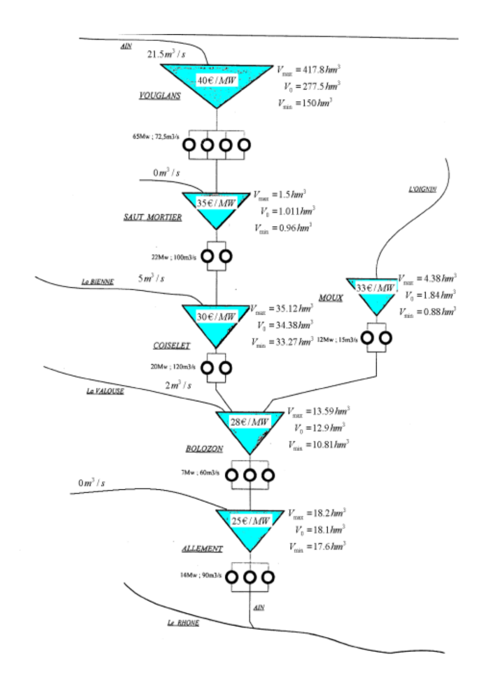
F4 : Robust

F5 : Max-p

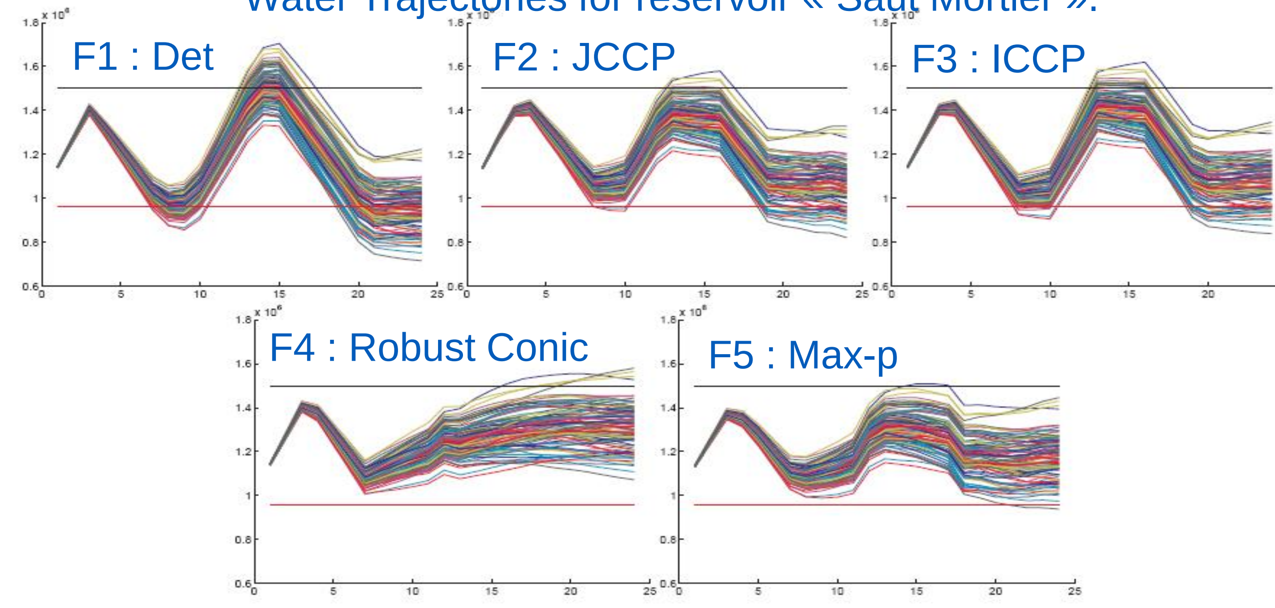
$$
\begin{array}{ll}
\max_{x \geq 0} & \varphi(x) := \mathbb{P}[a^r + A^r x \leq \eta \leq b^r + A^r x] \\
s.t. & Ax \leq b
\end{array}
$$

EXPERIMENT - THE HYDRO SUB-PROBLEM OF THE UCP

Numerical results



Water Trajectories for reservoir « Saut Mortier ».



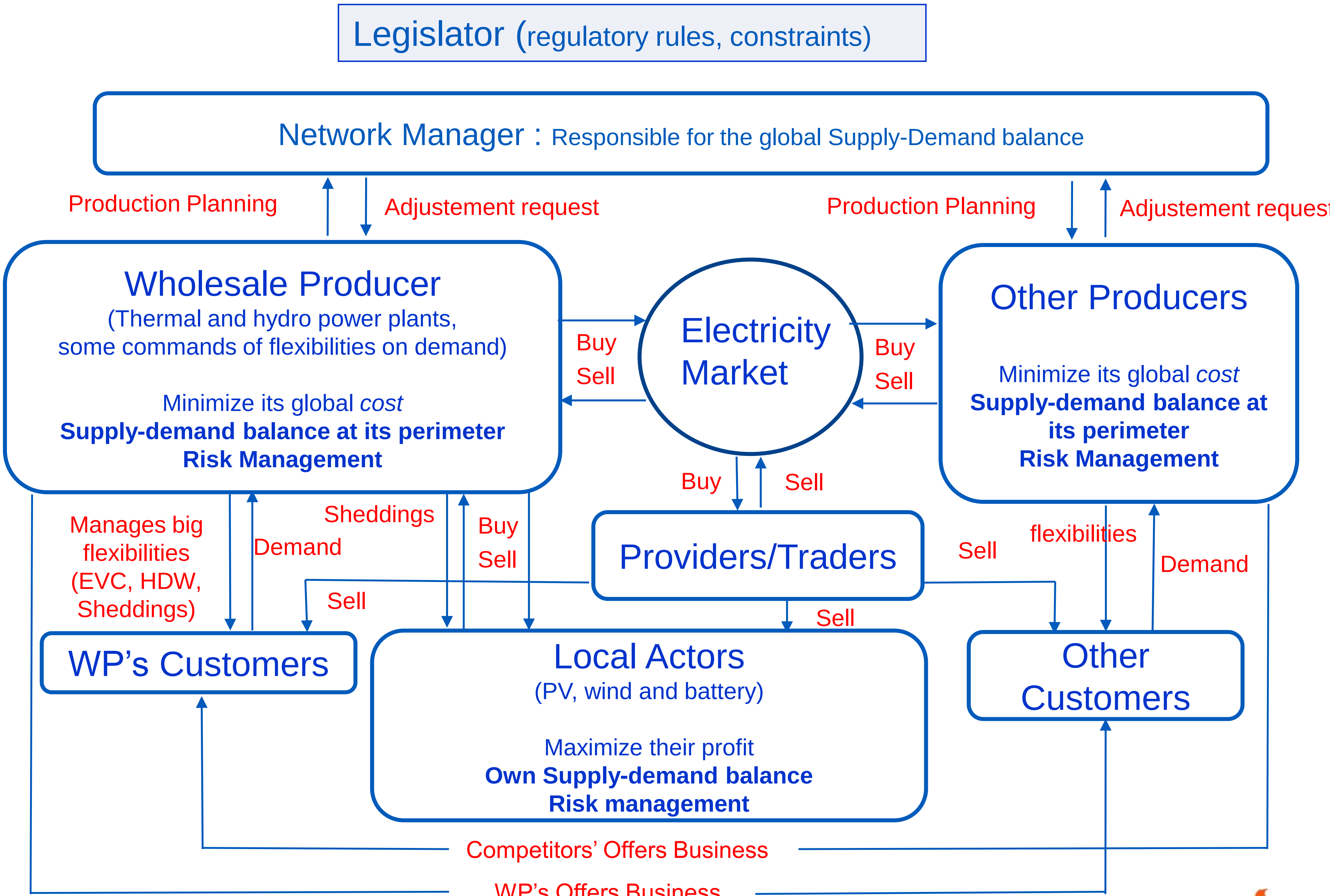
Inst.	Item / Problem	Det	JCCP	ICCP	Conic	Max-p
2	nbViolation	100	20	35	4	2
2	Cost (€)	$-1.0478e^5$	$-1.0340e^5$	$-1.0422e^5$	$-1.0282e^5$	$-9.9176e^4$

- The deterministic average-solution leads to almost sure violation of bounds
- JCCP gives a good trade-off between cost and robustness
- The Max-p solution is very robust, but very expensive

Which energy management problems

in a decentralized setting ?

THE DECENTRALIZED SETTING



RELATIONSHIPS BETWEEN WHOLESALE PRODUCER AND ACTORS

Wholesale Producer:
(Thermal and hydro power plants, some levers of flexibilities on demand)

Minimize its *costs*
Supply-demand balance at its perimeter

Commands on demand to facilitate sheddings via attractive pricing schemes

Buy / sell at wholesale producer

Local Actor :
(PV, wind and battery)

Maximize its profit

Own Supply-demand balance

Manages big flexibilities (EVC, HDW, Sheddings)

Demand

Customers

Optimization problem with uncertainty

$$
\begin{aligned}
\min_{\tilde{x}} \quad & \tilde{c}_1 \tilde{x} + \tilde{d}_1 \tilde{y} \\
s.t. \quad & \\
& \tilde{A}_1 \tilde{x} + \tilde{B}_1 \tilde{y} \leq \tilde{b}_1
\end{aligned}
$$

Wholesale producer : « Leader »
$\tilde{x}$: commands on its production and the demand generated by its customers

$$
\begin{aligned}
& \max_{\tilde{y}} \tilde{d}_2 \tilde{y} \\
s.t. \quad & \tilde{A}_2 \tilde{x} + \tilde{B}_2 \tilde{y} \leq \tilde{b}_2
\end{aligned}
$$

Local Actor : « Follower »
$\tilde{y}$: commands on its production and its own demand

THE WHOLESALE PRODUCER MANAGES FLEXIBILITIES/BATTERY

Joint optimization of production portfolio and flexibilities on demand

Supply

Classical production portfolio (nuclear, thermal, hydro)

Renewables Production manageable by using batteries

Demand

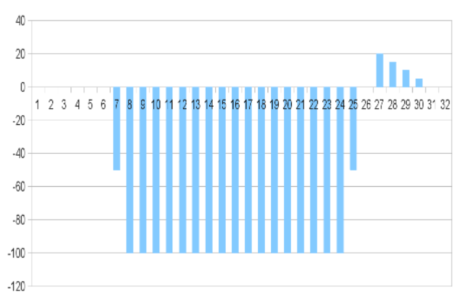
Three main flexibilities

Shedding Contracts

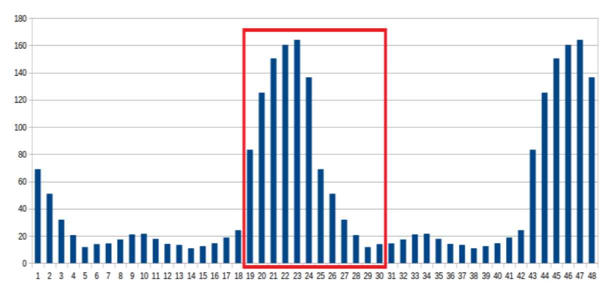
Electrical Vehicles Charging (EVC)

Hot Domestic Water (HDW)

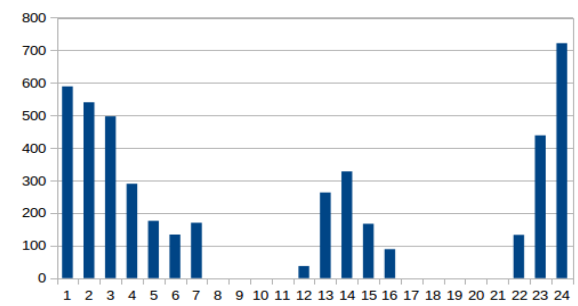
Shedding Profile



Electrical Vehicles Charging



Hot Domestic Water



Shedding on 24 h

Shift of EVC and HDW consumptions while respecting some constraints :
1) Respect of the maximal peak and 2) Energy conservation

2 slots : 09-17h et 18-06h

THE WHOLESALE PRODUCER MANAGES FLEXIBILITIES/BATTERY

Results (obtained with fictitious data unrelated with EDF)

Flexibilities

Mix	Nuclear	Coal	Fuel	Gas	Hydro	Wind	PV
2015	49.1 %	21.4 %	12.7 %	4.2 %	8.4 %	3.2 %	1.1 %
2030	28.2 %	15.7 %	11.2 %	5.6 %	12.8 %	16 %	10.6 %

Flexibility optimized	Saving on global cost (%)	
	2015	2030
Electrical Vehicles	0.26 %	0.24 %
Hot Domestic Water	0.62 %	2.59 %
Sheddings	0.04 %	0. %
The three	0.84 %	2.66 %

Simplified deterministic model

Battery management

We compare the 2 following situations :

- Situation 1 : The big producer manages the battery
- Situation 2 : The local player manages the battery

	Earning Situation 1	Earning Situation 2
For the big producer	1.45 %	0.72 %
For the Local Actor	0 %	4.09 %
For the both	1.41 %	0.82 %

It is generally more cost-effective to manage the battery by the big producer and give financial compensation to the local player… but
… the local player has personal interest in managing himself his battery.

SELF-CONSUMPTION

A new french 2016 law allows self-consumption

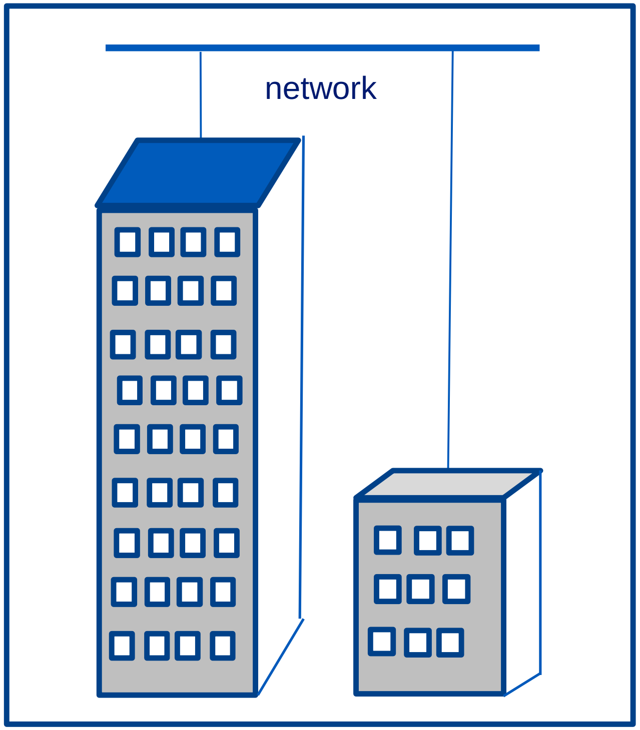
1 business building A

Load curves

Flexibilities : EVC, HDW

1 PV panel on the roof +battery

- Sharing the PV production
- Sell the excedent to building B



1 trading building B

Load curves

No PV panel

Problems

- Building A : each company who invested in the PV +battery aims at maximizing its profit
- Building B : each shop have to decide : buy electricity to the building A or not ?
- DSO : tarification for using the network
- Wholesale producer : estimating the residual demand induced by the self-consumption

TYPES OF PROBLEMS

Classical problems in a decentralized setting

→ **stochastic optimization**

Competition :

→ **game theory, equilibriums**

Relationships with customers :

→ **Tarification**
→ **Residual demands of prosumers**
→ **stochastic bi-level optimization**

CONCLUSIONS AND PERSPECTIVES

1) Even in "historical" situation of a centralized management, energy management optimization poses problems of extreme difficulty (large, uncertainty, continuous / discrete variables, big size)

2) The need to adapt to the new context of energy leads to new challenging optimization problems in interaction with economy and game theory. → Significant strengthening of the research effort.

EDF has a broad research program with the support of many academic institutions. Cooperation on these problems is possible via the two associated labs :

FIME : A research Lab in Finance for Energy markets

FINANCE FOR ENERGY MARKET RESEARCH CENTRE

A strong network in economics & applied mathematics

- **6** EDF research engineers
- **12** academic researchers
- **15** PhD students and postdocs in 2016
- **20** publications in peer-reviewed journals/year
- **4** High quality training sessions/year
- **4** partners

6 Published books since 2009

www.fime-lab.org

PGMO : the Gaspard Monge Program for Optimization

A strong network in optimisation, operational research & data science (+ education)

- **20** EDF research engineers
- **150** academic researchers
- **110** Laboratories contributing to projects
- **5**
- **35** publications in peer-reviewed journals/year
- **4** partners

www.fondation-hadamard.fr/en/pgmo

THANK YOU

APPENDIX

General Probabilistic Model

Theoretical Extension to Closed Loop Setting :

Based on exploiting the link between Joint and Individual Chance Constraints

Denote $\mathcal{E}_i = \{\xi \ : \ \langle A_i(\xi), x \rangle \geq b_i\}$

$$\mathbb{P}[A(\xi)x \geq b] = \mathbb{P}[\mathcal{E}_i] \prod_{k=1, k \neq i}^{m} \mathbb{P}[\mathcal{E}_k | \mathcal{E}_i \cap \cap_{j=1}^{k-1} \mathcal{E}_j]$$

→ Start with time step $i = 1$, sequential decision using conditional probabilities.

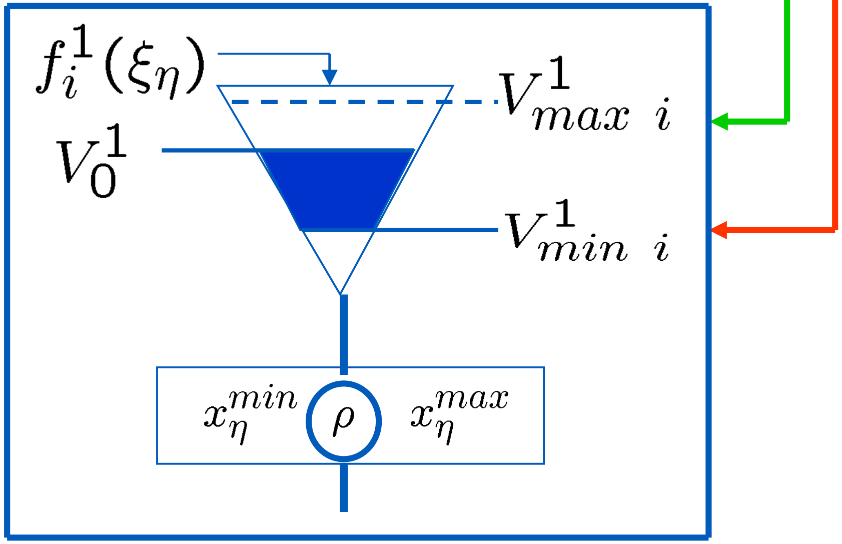
Probabilistic Model : Individual Chance Constraint

$$\begin{array}{ll} \min_x & c^t x \\ s.t. & \mathbb{P}[\langle A_i(\xi^\theta), x\rangle \geq b_i(\xi^\delta)] \geq \alpha_i, \ \ \forall i \\ & \mathbb{P}[\langle H_i, x^\eta\rangle \leq h_i^{min}(\xi^\eta)] \geq \beta_i, \ \ \forall i \\ & \mathbb{P}[\langle H_i, x^\eta\rangle \geq h_i^{max}(\xi^\eta)] \geq \beta_i, \ \ \forall i \\ & \langle P_i, x\rangle \leq h_i, \ \ i = 1, ..., m \\ & x \geq 0 \end{array}$$

$$h_i^{min}(\xi^\eta) = v_0 + \sum_{i0=1}^{i} f_{i0}(\xi^\eta) - v_i^{min}$$

$$h_i^{max}(\xi^\eta) = v_0 + \sum_{i0=1}^{i} f_{i0}(\xi^\eta) - v_i^{max}$$

$$H_{ij} = 1\!\!\mathrm{I}_{j \leq i}$$



Available Information

historical data

$$a_{ij}(\xi_\alpha) \in [a_{ij}(\xi_\alpha)_{min},\ a_{ij}(\xi_\alpha)_{max}]$$
$$b_i(\xi_\delta) \in [b_i(\xi_\delta)_{min},\ b_i(\xi_\delta)_{max}]$$

Exploiting minimal information on random processes :

$$a_{ij}(\xi)_{min},\ a_{ij}(\xi)_{mean},\ a_{ij}(\xi)_{max}$$

Hypothesis : Independence of

$$a_{ij}(\xi_\alpha),\ b_i(\xi_\delta)$$

Convex Approximation

Proof : Based on Hoeffding's Theorem :
for independent and bounded random variables $X_1, ..., X_m$, noting $S = \sum_{i=1}^{m} X_i$, the individual chance constraint is bounded as follows:

$$\forall \tau \geq 0,\ \mathbb{P}\left[S \geq \mathbb{E}[S] + \tau\right] \leq \exp\left(\frac{-2\tau^2}{\sum_{i=1}^{m}(\overline{X}_{ij} - \underline{X}_{ij})^2}\right).$$

We define a matrix $A' = \begin{pmatrix} A & -b \end{pmatrix}$ of dimension $(m, n+1)$ and the vector $x' = \begin{pmatrix} x & x_{n+1} \end{pmatrix}$ with $x_{n+1} = 1$.
We pose $X_{ij} = a^{'}_{ij} x^{'}_{j}$. $\square$

$$\longrightarrow \quad \mathbb{P}[\sum_{j=1}^{n} a_{ij} x_j \geq b_i] \Longleftrightarrow \mathbb{P}[\sum_{j=1}^{n+1} a^{'}_{ij} x^{'}_{j} \geq 0]$$

Computing the solution : solve a SOCP

ICCP : Individual Chance Constraint Program

$$\begin{aligned} \min \quad & c^t x \\ s.t. \quad & \mathbb{P}[\langle A_i(\xi_\theta), x\rangle \geq b_i(\xi_\delta)] \geq \alpha_i \quad i = 1, ..., m \\ & Px \leq h \\ & x \geq 0 \end{aligned}$$

SOCP : Second Order Conic Program

$$\begin{aligned} \min_x \quad & c^t x \\ s.t. \quad & \left\| \tilde{A}_l x + \tilde{b}_l \right\|_2 \leq \tilde{f}_{l^t} x + \tilde{d}_l, l \in (1, L) \end{aligned}$$

Resolution (polynomial complexity) : Interior Points Method

Equivalent SDP representation

$$\begin{aligned} \min_x \quad & c^t x \\ s.t. \quad & \begin{bmatrix} (\tilde{f}_l^t x + \tilde{d}_l) I & \tilde{A}_l x + \tilde{b}_l \\ (\tilde{A}_l x + \tilde{b}_l)^t & \tilde{f}_l^t x + \tilde{d}_l \end{bmatrix} \succeq 0, \;\; l \in (1, L) \end{aligned}$$

Resolution of Joint CCP Problem

- The joint CCP can be shown to be convex.

- Step 1 : Gradient can be computed by repeated use of Genz' code.

Corollary 1 *Let ξ be a Gaussian Random variable of dimension n. Let x, A,B,a,b be vectors and matrices of appropriate dimension. Now consider the mapping $\varphi : x \mapsto \mathbb{P}[a + Ax \leq \xi \leq Bx + b]$. We have:*

$$\nabla\varphi = \nabla_a F_\xi(a,b)^\mathsf{T} A + \nabla_b F_\xi(a,b)^\mathsf{T} B$$
$$\triangle\varphi = A^\mathsf{T} \triangle_{aa} F_\xi(a,b)A + A^\mathsf{T} \triangle_{ab} F_\xi(a,b)B + B^\mathsf{T} \triangle_{ba} F_\xi(a,b)A + B^\mathsf{T} \triangle_{bb} F_\xi(a,b)B.$$

- Step 2 : We can apply a cutting planes algorithm to solve the problem

The Hydro Sub-Problem : Solving the Joint CCP Problem

- Step 1 : gradient of a joint chance constraint

Theorem 1 *Assume that* $\xi \sim \mathcal{N}(\mu, \Sigma)$ *with some positive definite covariance matrix* Σ. *Then, for* $i = 1, \ldots, n$,

$$\frac{\partial}{\partial b_i} F_\xi(a, b) = f_{\xi_i}(b_i)\, F_{\tilde{\xi}(b_i)}(\tilde{a}, \tilde{b}) \tag{2}$$

$$\frac{\partial}{\partial a_i} F_\xi(a, b) = -f_{\xi_i}(a_i)\, F_{\tilde{\xi}(a_i)}(\tilde{a}, \tilde{b}). \tag{3}$$

Here, f_{ξ_i} *is as in Lemma 1,* $\tilde{\xi}(b_i), \tilde{\xi}(a_i)$, *are* $n-1$*-dimensional random vectors distributed according to* $\tilde{\xi}(b_i), \tilde{\xi}(a_i) \sim \mathcal{N}(\hat{\mu}, \hat{\Sigma})$ *such that* $\hat{\mu}$ *results from the vector* $\mu + \sigma_{ii}^{-1}(b_i - \mu_i)\sigma_i$ *(in case of* b_i*) or from the vector* $\mu + \sigma_{ii}^{-1}(a_i - \mu_i)\sigma_i$ *(in case of* a_i*) by deleting component* i *and* $\hat{\Sigma}$ *is defined as in Lemma 1. Moreover* $\tilde{a}$ *and* $\tilde{b}$ *result from* a *and* b *by deleting the respective component* i.

Resolution of Joint CCP Problem

- Step 2 : cutting planes algorithm

1. Let x_0 be the solution of the "average" problem, x_s a Slater point. Set $A_0 = A$, $b_0 = b$ and $k = 0$.

2. Find λ^* such that $x_k^* = (1 - \lambda^*)x_k + \lambda^* x_s$ and $\left|\varphi(x_k^*) - p\right| < \varepsilon$.

3. Add constraint $-\nabla\varphi(x_k^*)^\top x \leq \varphi(x_k^*) - \nabla\varphi(x_k^*)^\top x_k^* - p$ to the matrix.

4. Solve $\min_x c^\top x; s.t. A_k x \leq b_k$ to find x_{k+1}.

5. If $\varphi(x_{k+1}) > p - \delta$ then stop, else move $k = k + 1$ and go in step 2.

Probabilistic/Robust Approaches

F4 : Robust Conic

Uncertainty set : find some set $\mathcal{E}_p$ such that $\mathbb{P}[\eta \in \mathcal{E}_p] \approx p$ such as

$$\mathcal{E}_p = \left\{x : x^\top \Sigma^{-1} x \leq n + \Phi^{-1}(p)\sigma_C\right\}$$

with
$$\sigma_C = \sqrt{\sum_{i=1}^{n} \mathbb{E}\left(y_i^4\right) - n}$$
$$y = L^{-1}\eta$$
$$\Sigma = LL^\top.$$

Solve :
$$\begin{aligned} \min_{x \geq 0} \quad & c^\top x \\ s.t. \quad & Ax \leq b \\ & a^r + A^r x \leq \inf \mathcal{E}_p \\ & b^r + A^r x \geq \sup \mathcal{E}_p \end{aligned}$$

$\inf \mathcal{E}_p$: $largest\ vector\ x^i \in \mathbb{R}^n\ having\ x^i \leq y\ \forall y \in \mathcal{E}_p$