Catastrophes and Expected Marginal Utility – How The Value Of The Last Fish In A Lake Is Infinity And Why We Shouldn't Care (Much)

Eric Nævdal[1]
The Ragnar Frisch Centre for Economic Research
University of Oslo
Gaustadallén 21, 0349 Oslo, Norway
e-mail: eric.navdal@frisch.uio.no

Abstract

Catastrophic risk is currently a hotly debated topic. This paper contributes to this debate by showing two results. First it shown that the value function in dynamic optimization can have an infinite derivative at some point even if the model specification has functional forms that are finite and without infinite derivatives. In the process it is shown that standard phase diagrams used in optimal control theory contain more information than generally recognized. Second we show that even if the value function has an infinite derivative at some point, it is not correct that this point should be avoided in finite time at almost any cost. The results are illustrated in a simple linear-quadratic fisheries model, but proven for a more general class of growth functions.

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1. Introduction

Weitzman's seminal (2009) contribution has been much debated since it's publication. The "dismal theorem" states that under certain conditions, the process of gathering data about a stochastic process leads to a situation where expected marginal utility explodes. Weitzman is careful to state that his result does not depend on "a mathematically illegitimate use of the symbol $+\infty$," but arises naturally and that criticism of his results by "somehow discrediting this application of expected utility on the narrow grounds that infinites are not allowed in a legitimate theory of choice under uncertainty" is unfounded. This paper will show that infinities are most certainly allowed in such a theory as they under certain conditions turn up endogenously in models where all functional forms have everywhere finite derivatives. Thus the data gathering process applied in Weitzman (2009) is not required to generate infinite shadow prices. The dismal theorem should apply to any process that generates a probability of infinite marginal utility or infinite shadow prices in the event of an catastrophe.

The question is then if infinite shadow prices or infinite marginal utility actually matter. Weitzman himself argues that "The burden of proof if in climate change is presumably upon whoever calculates expected discounted utilities without considering that structural uncertainty might matter more than discounting or pure risk. Such middle-of-the-distribution modeler should be prepared to explain why the bad fat tail of the posterior predictive PDF does not play a significant role in climate-change CBA when it is combined with a specification that assigns high disutility to high temperatures." This statement must be qualified. The dismal theorem is about expected *marginal* utilities and not the *level* of disutility. But even so, the dismal theorem seems to indicate that getting into a situation where marginal utility is infinity should be avoided at almost any cost. Indeed, when Inada conditions specify that functions evaluated at zero have infinite derivatives, the purpose is to ensure that a model economy converges to a interior steady state. This property to some extent carries over to stochastic models as it has been known since Brock & Mirman (1972) that optimal paths converge to a uniquely non-trivial stationary solution and that this result depends on imposing Inada-conditions. Kamihigashi (2006) has shown that if Inada conditions are not imposed, there will be almost sure convergence to zero. One could therefore be excused for thinking that imposing Inada conditions guarantee against it ever being optimal to reach a state with infinite shadow prices, typically a state where the amount of some valuable variable is equal to zero.[2] It is shown below that this turns out to be wrong in models where a catastrophe is defined as a total collapse in the state variable. The Inada conditions are sufficient to eliminate the possibility of reaching a state with infinite shadow prices through an *incremental* reversible process, but such results do not apply to the kind of major shocks we associate with large disruptions such as the rapid disintegration of the Western Antarctic Ice Sheet or disruption of the Thermohaline Circulation, Nævdal (2006), Nævdal and Oppenheimer (2007).

The realism of Weitzman's results have be challenged by e.g. Nordhaus (2011), Pindyck (2011) and Costello *et al* (2010). These authors have argued that minor realistic changes in the assumptions invalidates the dismal theorem and argue that the dismal theorem is built on unrealistic assumptions. The claim made in the present paper is that alternative realistic assumptions can generate the assumptions underlying the dismal theorem, but that the theorem often does not matter for optimal policy prior to large catastrophes as the ex post shadow price will fully or partially cancel out of optimality conditions or so that the effect of the infinite shadow prices is limited.

The remaining paper is organized as follows. Section 2 shows that an infinite shadow price turns up in a very standard fisheries model where instantaneous utility and the equation of motion is linear in the control variable. Instantaneous utility is assumed not to depend on the state variable and the equation of motion is quadratic in the state variable. It is shown that even this model generates an infinite shadow price on the state variable at zero. It is further shown that even if there is an infinite shadow price on the fish stock when the stock is zero, it may be optimal to accept that the fishery will collapse with probability one even if a deterministic model would never produce this

[2] Indeed, Mirman and Zilcha (1976) provide an example where consumption is not bounded away from zero even if Inada-conditions are imposed.

outcome. Section 3 shows that the results from the fisheries model can be generalized to a class of problems characterized by being regenerative in the sense that if the state variable is zero, so is the growth. Further it is shown that regardless of how infinite shadow prices are generated, it follows from the optimality conditions that the infinite value of shadow prices in certain states are not of great importance in deciding whether one should allow that such states are reached with positive probability.

2. A simple fisheries model

Clark and Munro (1975) presented a model that gave the theory of renewable resources a proper capital theoretic foundation and has been a cornerstone of resource economics ever since. In their model they derive the importance of the relationship between the social discount rate and the intrinsic growth rate. Here the simplest version of that model is applied. The analysis is in two stages. First a deterministic model is analyzed and it is shown that shadow price of the fish stock evaluated at zero is in fact infinity. Second, the deterministic model is a building block in a model where we analyze the possibility of a complete collapse in the fish stock. It is shown that even if the intrinsic growth rate is larger than the social discount rate, in the presence of catastrophic risk it is optimal to allow the stock of fish to become extinct with probability 1. The model is similar to Polasky *et al* (2011), but differ in the type of shock that is analyzed. This is the simplest possible model of the benefits generated by a fishery. Many, if not all, fisheries provide additional benefits such as existence values and ecosystem values that are not captured in the present model. They should certainly be incorporated before allowing a fishery to become extinct.

2.1. The marginal value of the last fish in a lake is infinity

The simplest version of the textbook dynamic fishery model assumes an exogenous price of fish, p, harvesting, $h \in [0, \bar{h}]$, is assumed to be costless and the equation of motion is given by:

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - h \tag{1}$$

Here r is the intrinsic growth rate of the stock and K is the carrying capacity. This leads to the optimization problem

$$V\left(x\left(0\right)\right) = \max_{h\left(t\right)} \int_0^\infty ph\left(t\right)e^{-\rho t}dt \tag{2}$$

subject to (1) and $x(0)$ given. This model is well known and easy to analyze. The Hamiltonian is given by:

$$H = ph + \mu\left(rx\left(1 - \frac{x}{K}\right) - h\right) \tag{3}$$

This leads to the following optimality conditions:

$$\begin{aligned} &p > \mu \rightarrow h = \overline{h} \\ &p = \mu \rightarrow h = rx\left(1 - \frac{x}{K}\right) \\ &p < \mu \rightarrow h = 0 \end{aligned} \tag{4}$$

$$\dot{\mu} = \rho\mu - r\mu\left(1 - \frac{2x}{K}\right) \tag{5}$$

Combining (4) and (5) with the appropriate transversality conditions determines the optimal program.[3] The long run equilibrium for this program is:

$$h_{ss} = \frac{K\left(r^2 - \rho^2\right)}{4r},\ x_{ss} = \frac{K\left(r - \rho\right)}{2r},\ \mu_{ss} = p \tag{6}$$

Assuming that $r > \rho$, it is always optimal to have positive harvesting and a positive stock in steady state. The behavior of the optimal program is summarized in the phase diagram in Figure 1. We are mostly concerned with the stable manifold which represents the optimal path of the fishery through (x, μ)-space. Any optimal path from an arbitrary starting point $x(0)$ starts on this manifold and converges to the steady state. The manifold can be interpreted as a function where for any x , μ is the corresponding

[3] Strictly speaking, when $\mu = p$ one is indifferent between different values of h. We here follow the common practice to chose the particular h that sets $\dot{x} = 0$ so that no further jumps in h are required.

shadow price. We shall denote this function $\mu(x)$. Obviously $\mu(x)$ is the derivative of the value function $V(x)$. Thus the value function can be found by integrating $\mu(x)$ over $[0, x]$ and can be illustrated in the phase diagram as the area below the stable manifold.[4] For our purposes, the most important thing to note is that $\mu(x)$ appears to grow without limit as $x \to 0$.

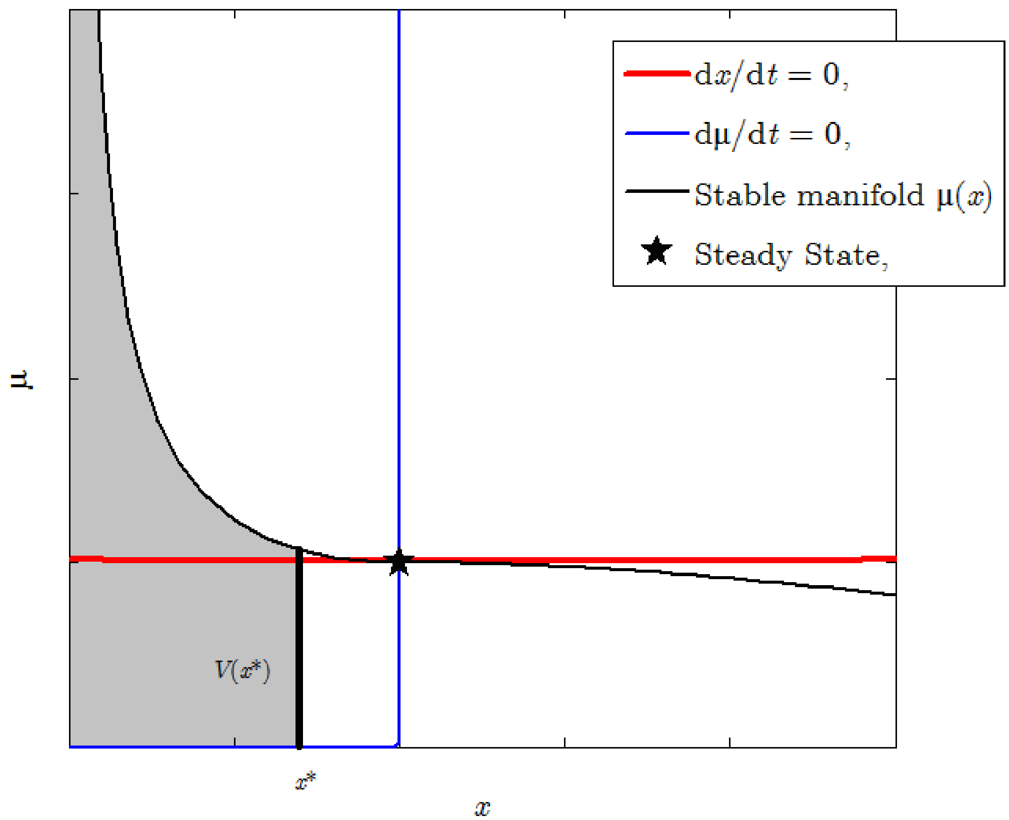

Figure 1. Phase diagram for the simple fisheries model. The stable manifold shows combinations of μ and x along an optimal path and is the derivative of the value function. The value function can be computed for any x^* by computing the area under $\mu(x)$ over the interval $[0, x^*]$ as illustrated by the shaded area.

[4] The interpretation of the stable manifold as the derivative of the value function and the area below it as the value function seems to be unrecognized among economists. An informal survey of close to 20 respected resource economists revealed that none were aware of this interpretation.

Indeed, in the present model one can calculate $\mu(x)$ over the interval $(0,\ x_{ss}]$ and get that:[5]

$$\mu\left(x\right) = \frac{pK^2}{4r^2} x^{\frac{\rho - r}{r}} \left(K - x\right)^{-\frac{r+\rho}{r}} \left(r + \rho\right)^{\frac{r+\rho}{r}} \left(r - \rho\right)^{\frac{r-\rho}{r}} \tag{7}$$

This expression clearly satisfies $\lim_{x \to 0} \mu\left(x\right) = \infty$ and in standard economic parlance, the value of the last fish in the lake is infinity. Of course, in the present model one will never experience this along an optimal path. The property $\lim_{x \to 0} \mu\left(x\right) = \infty$ is simply the way that optimization ensures that we never allow the fish stock to become extinct as long as $r > \rho$, regardless of the values of the other parameters. Having calculated (7) it is straightforward to show that for $x \leq x_{ss}$ it holds that the value function is given by:

$$V\left(x\right) = \left(r + \rho\right)^{\frac{r+\rho}{r}} \left(r - \rho\right)^{\frac{r-\rho}{r}} \frac{pK}{4r\rho} \left(\frac{x}{K - x}\right)^{\frac{\rho}{r}} \tag{8}$$

Clearly, $V(0) = 0$. $V(x)$ and $\mu(x)$ are drawn in Figure 2 for all $x \in [0,\ K]$. The maximum value the fishery can provide is given by $V(K)$, unless there is an biblical event pushing the stock level above K.[6]

[5] Equation (7) is a special case of a more general solution given in Equation (15) below. The expression for $\mu(x)$ can also be calculated over the interval $[x_{ss},\ K]$, but is rather messy.

[6] I.e. a rain of fish.

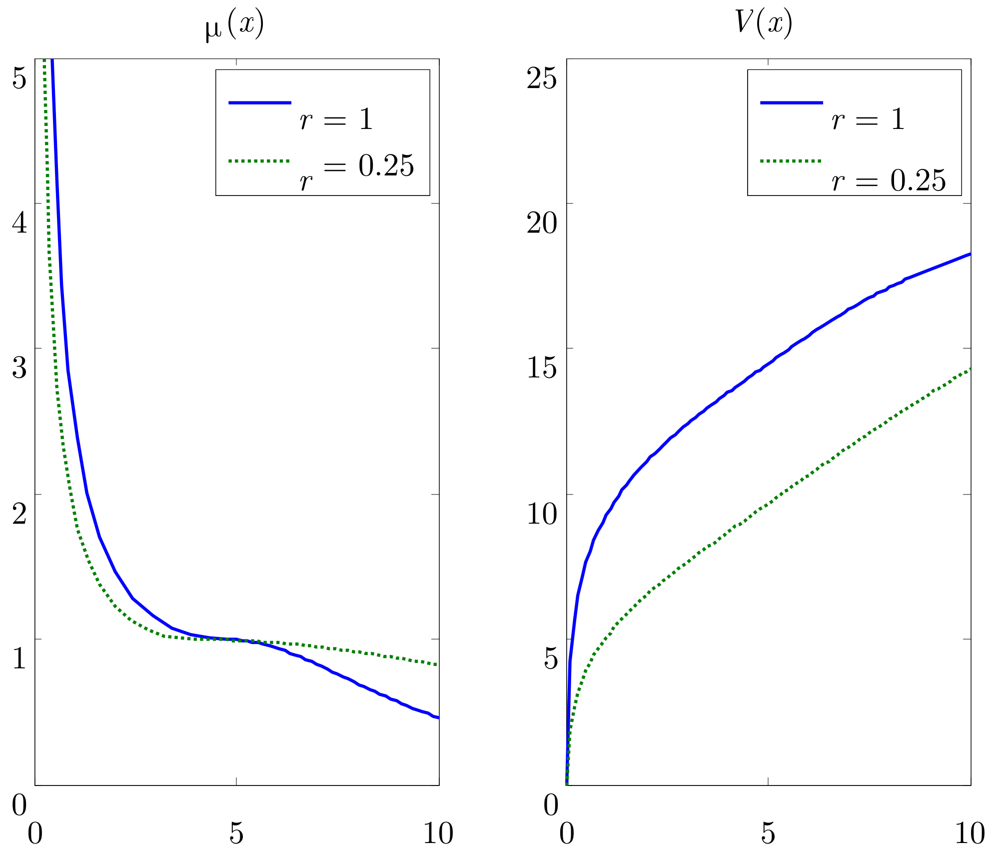

Figure 2. The shadow price and the value function as functions of *x*.

2.2. Why we don't care (much)

We postpone for the moment a more technical treatment and ask what implications it has that $\lim_{x \to 0} \mu(x) = \infty$. In this deterministic setting the only implication is that regardless of parameter values, as long as $r > \rho$, it is never optimal to let the fish stock become extinct. The interesting thing for our purposes is that $\lim_{x \to 0} \mu(x) = \infty$ does not imply that we will avoid the state $x = 0$ at any cost. Consider e.g. the following scenario where, if we allow the stock of fish go below a certain level, we introduce the possibility of extinction. Let τ be the time at which the catastrophe occurs, and let the catastrophic effect be that the fish becomes extinct. That is:[7]

$$x\left(\tau^{+}\right)-x\left(\tau^{-}\right)=-x\left(\tau^{-}\right) \tag{9}$$

The probability of extinction at any point in time is assumed given by:

$$\lim_{dt\to 0}\frac{\Pr\left(\tau\in\left[t,t+dt\right[\mid \tau\geq t\right)}{dt}=\lambda\left(K-x\left(t\right)\right)\geq 0,\ \lambda\left(0\right)=0 \tag{10}$$

$\lambda(\cdot)$ is assumed to be continuous, increasing and differentiable.
If the stock is harvested to levels permanently below K, clearly the fish stock will become extinct at some point in time with probability one. Are we to avoid this outcome at all possible costs? Clearly not. If we allow some fishing, we get a least some profit out of the fishery before it becomes extinct. Thus our profit from fishing is something + zero. If we do not, we have zero profits. The possibility of ending up in a state where the derivative of the value function is zero does not preclude us from making a quick buck.

Formally, what we have done is to divide state-space into two subsets. One where there is risk and one where there is not. Margolis and Nævdal (2007) has shown that unless the non-risky subset contains an attractor, it will always be optimal to enter the risky subset unless there is a discontinuity in the *derivative* of the hazard rate at the boundary between the two sets. A discontinuity in the derivative of the hazard rate at the boundary is a necessary, but not sufficient condition for staying in the non-risky subset. This result holds as long as the consequences of the catastrophe is not a pay-off of $-\infty$. In our fisheries example, we have assumed that the derivative of hazard rate is continuous at the boundary and it is therefore optimal to accept that the fishery will become extinct with probability 1.

[7] Here $x(\tau^{+})=\lim_{t\to\tau^{+}}x\left(t\right)$ and $x(\tau^{-})=\lim_{t\to\tau^{-}}x\left(t\right)$

3. A General Analysis

3.1. Why infinite marginal utility can turn up endogenously

In the fisheries example it was shown that the marginal value of the last fish is infinity. Here we explain that it is in fact to be expected that this will happen in a large number of applications. Consider a general autonomous optimal control problem with infinite time horizon where instantaneous utility is $F(u, x)$, the stock grows according to $\dot{x} = f(x, u)$, where $A = \left[0, a\right) \subseteq \left[0, \infty\right)$ is the feasible set of x values and initial conditions $x(0)$ are given. The value of u that maximizes the Hamiltonian can be written $u(x, \mu)$ where μ is the co-state variable or shadow price on x. Necessary conditions may then be written:

$$\dot{\mu} = r\mu - \mu F_x'\left(x, u\left(x, \mu\right)\right) - \mu f_x'\left(u\left(x, \mu\right), x\right) \tag{11}$$

and:

$$\dot{x} = f\left(x, u\left(x, \mu\right)\right) \tag{12}$$

We shall assume the existence of a steady state value in A and denote the steady state values of x and μ as x_{ss} and μ_{ss} respectively. We wish to calculate the stable manifold over A. That is for any given x we want to calculate the value of μ that is the shadow price of x evaluated at that given x. A conceptually straight forward method for doing this is to solve the following differential equation:

$$\frac{\dot{\mu}}{\dot{x}} = \frac{\mathrm{d}\mu}{\mathrm{d}x} = \frac{r\mu - F_x'\left(x, u\left(x, \mu\right)\right) - \mu f_x'\left(x, u\left(x, \mu\right)\right)}{f\left(x, u\left(x, \mu\right)\right)}, \ \mu\left(x_{ss}\right) = \mu_{ss} \tag{13}$$

Actually computing this differential equation can be tricky. Analytical solutions can only be found for a few special cases, such as (7), and numerical solutions must recognize that evaluating the differential equation at $\mu\left(x_{ss}\right) = \mu_{ss}$ implies evaluating a "0/0" expression. See Judd (1998), chapter 10.7, for workarounds. The expression in (13) will be quite important for what follows.

I claim that there are three cases that may lead to $\lim_{x\to 0}\mu\left(x\right)=\infty$ as they all imply that $d\mu/dx_{x\to 0}=\infty$:[8]

1. $F_u'\left(0,0\right)=\infty$. This is case is well known and is the case that is used in the dismal theorem debate.
2. The next case is when the numerator in (13) is infinite when evaluated at $x=0$. This case is also well known and rather intuitive.
3. The last possibility is when the denominator goes to zero as x goes to zero. That is when $f\left(0,u\left(0,\mu\right)\right)=0$.

Case 1 and 2 require that the model uses specific functional forms with infinite derivatives. Case 3 requires no such thing and would occur quite naturally in a number of settings. Most models of renewable resources has an assumption built in that growth in the resource is zero when the stock is zero. Aggregate macro production functions assume that capital is required to create more capital. There are surely other examples as well.

The main point here is that there are a number of naturally occurring processes that have a self-generative nature and whenever one does dynamic economic analysis of them, one must accept that there is a possibility that the shadow price on that resource goes to infinity as the resource goes to zero.

To show that Case 3 may generate infinite shadow prices we prove a slightly more general version of the fisheries model above.

Proposition 1.

Assume that a regulator wants to maximize (2), given the growth function:

$$\dot{x}=g\left(x\right)-h,\ g\left(0\right)=0,\ \rho<g'\left(0\right)<\infty \tag{14}$$

Assume further that there exists a steady state given by (x^*, p). Then $\mu(x)$ is given by:

[8] $d\mu/dx_{x\to 0}=\infty$ is in itself neither a necessary or a sufficient condition for $\lim_{x\to 0}\mu(x)=\infty$.

$$\mu\left(x\right) = p\frac{g\left(x^*\right)}{g\left(x\right)}\exp\left(-\int_x^{x^*}\frac{\rho}{g\left(y\right)}dy\right) \quad (15)$$

Proof:

As instantaneous utility is linear in the control, we get that in the interval [0, x^*) we have that $h = 0$, so for $x \leq x^*$ we can write:

$$\frac{\dot{\mu}}{\dot{x}} = \frac{d\mu}{dx} = \frac{\rho\mu - \mu g'\left(x\right)}{g\left(x\right)},\ \mu\left(x^*\right) = p \quad (16)$$

This is a separable differential equation and the solution is shown in the appendix to be given by (15).

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Proposition 2.

Assume that $\rho < g'\left(0\right)$. Then $\lim_{x\to 0}\mu\left(x\right) = \infty$.

Proof:

It is well known that as long as $\rho < g'\left(0\right)$, $\mu(x) > 0$ for all x. There are two cases to consider. If the integral in (15) converges, then the proof is trivial as $g(0)$ is zero. If the integral does not converge, we have a "0/0" expression which can be evaluated using L'Hôpital's rule. It is shown in the appendix that application of L'Hôpital's rule yields the following expression:

$$\lim_{x\to 0}\mu\left(x\right) = \frac{\rho}{g'\left(0\right)}\lim_{x\to 0}\mu\left(x\right) \quad (17)$$

As $\rho < g'\left(0\right)$, this is only possible if $\lim_{x\to 0}\mu\left(x\right) = \infty$.

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This result is quite strong. In a process where instantaneous utility is linear in the control, the shadow price goes to infinity regardless of the shape of the growth function as long as $g(0) = 0$. Having established $\lim_{x\to 0}\mu\left(x\right) = \infty$ in a simple model, we can easily extend the result to more general models. E.g. if instantaneous utility can be written e.g. as $pu + H(x)$ where $0 < H'(x) < \infty$, then (16) becomes

$$\frac{\dot{\mu}}{\dot{x}} = \frac{d\mu}{dx} = \frac{\rho\mu - H'(x) - \mu g'(x)}{g(x)} \tag{18}$$

This expression is steeper than (16) as x goes to zero, which implies that also in this case $\lim_{x\to 0}\mu(x) = \infty$. A similar argument can be made for the case where the control h is positive for all $x > 0$ but goes to zero as x goes to zero.

3.2. Why we still don't care (much)

Having argued that infinite shadow prices turn up quite often and endogenously, the question is then if we should be willing to use large resources to avoid a probability that we experience a state of the world with such a situation. The answer is no, and the reason can be found in the first order conditions for optimization problems with catastrophes. Note that the arguments presented below holds for *any* case where there is a infinite shadow prices and is not limited to the case where we encounter divide by zero in (18). Thus the arguments below apply also to cases where $F_u'(0,0) = \infty$.

We consider an arbitrary optimal control problem with the notation from 3.1. The problem has two phases, one before and one after a catastrophe. The problem after the catastrophe is a simple deterministic control problem with value function $V(x)$ and shadow price $\mu(x) = V'(x)$. We impose that $\lim_{x\to 0}\mu(x) = \infty$. Without loss of generality we assume $\mu(x) \geq 0$ so that the state variable is a desirable commodity. We specify a hazard process such that the hazard rate at any point in time is given by:

$$\lambda = \lambda(x,t) \tag{19}$$

Note that we do not at this point exclude the possibilities that λ depend on only x or only t or is a constant for that matter. Denote the point in time that the shock occurs as τ and the shock that occurs is that x experiences a jump given by $g(x)$. Thus $x(\tau^+) = x(\tau^-) + g(x(\tau^-))$. Using optimality conditions found in Seierstad (2009), page 130, we can phrase the optimality conditions in the terms of an Risk-Augmented Hamiltonian.

$$H\left(u,x,\mu,z\right)=F\left(x,u\right)+\mu_r f\left(u,x\right)+\lambda\left(x,t\right)\left(V\left(x+g\left(x\right)\right)-z\right) \qquad (20)$$

Here z is the value function at time t conditional on the catastrophe not having occurred over the time interval $[0,\ t]$. $V(x)-z$ is thus the cost of the catastrophe and here assumed to be negative. μ_r is the shadow price prior to the catastrophe occurring. Denoting the optimal control as u^*, optimality conditions are:[9]

$$u^*=\arg\max_u H\left(u,x,\mu_r,z\right) \qquad (21)$$

$$\begin{aligned}\dot{\mu}_r &= \rho\mu_r - F_x'\left(x,u^*\right)-\mu_r f_x'\left(x,u^*\right)\\ &\quad -\lambda\left(x,t\right)\left(\mu_r-\mu\left(x+g\left(x\right)\right)\left(1-g'\left(x\right)\right)\right)-\lambda_x'\left(x,t\right)\left(z-V\left(x+g\left(x\right)\right)\right)\end{aligned} \qquad (22)$$

$$\dot{z}=rz-F\left(x,u^*\right)-\lambda\left(x,t\right)\left(V\left(x+g\left(x\right)\right)-z\right) \qquad (23)$$

Recall that one interpretation of $\dot{\mu}$ is that for a given value of x, the abs($\dot{\mu}$) is a measure of how urgently we would like to move to other locations in state space. In the present context, if $\dot{\mu}$ is smaller than 0, this implies that we are at a location in state space where the probability and/or the consequences of a catastrophe are severe and we wish to move to safer grounds. Formally, the possibility of disaster affects the optimality conditions through (22) and (23). Equation (22) is the differential equation for the co-state variable. The first three terms of the right hand side (22) are the same as in a deterministic control problem. The two remaining terms are:

$$-\underbrace{\lambda\left(x,t\right)\left(\mu_r-\mu\left(x+g\left(x\right)\right)\left(1-g'\left(x\right)\right)\right)}_{\text{Effect caused by different ex post and ex ante shadow price}}-\underbrace{\lambda_x'\left(x,t\right)\left(z-V\left(x+g\left(x\right)\right)\right)}_{\text{Effect caused by different ex post and ex ante value functions}} \qquad (24)$$

These two terms are the modification that catastrophic risk requires to the differential equation for μ. The term $-\lambda\left(x,t\right)\left(\mu_r-\mu\left(x+g\left(x\right)\right)\left(1-g'\left(x\right)\right)\right)$, which is the

[9] Transversality conditions must also me checked. They are similar to transversality conditions in deterministic control theory.

probability of a disaster at any point in time multiplied by the difference between the ex post and ex ante shadow price, determines how $\dot{\mu}_r$ respond to differences in the shadow price before and after a catastrophe. The last term, $-\lambda'_x(x,t)(z - V(x + g(x)))$ show how $\dot{\mu}_r$ respond to differences in the absolute levels of welfare before and after a catastrophe. Note that the only place where *ex post* shadow price $\mu(x)$ enters optimality conditions is in (22) where there is a term $\mu(x + g(x))(1 + g'(x))$. Now let us consider the consequences of a catastrophe. The straight forward way to do so is to stipulate that x collapses to zero. Thus $g(x) = -x(\tau^-)$ and $g'(x) = -1$. Examining $\mu(x + g(x))(1 - g'(x))$ shows us that $\mu(x + g(x))(1 - g'(x))$ is on the form "$\infty\times 0$." We therefore examine a limit where we let $g(x) = -\alpha x(\tau^-)$ and $g'(x) = -\alpha$ and study the behavior as $\alpha \to 1$. We have three possible outcomes:

$$\lim_{\alpha \to 1} \mu\left(x\left(\tau^-\right) - \alpha x\left(\tau^-\right)\right)\left(1-\alpha\right) = \begin{cases} \infty \\ d \in \mathbb{R}/0 \\ 0 \end{cases} \tag{25}$$

If the expression in (25) goes to infinity then we would clearly expend substantial resources to avoid a probability of a shock. If it is some constant d, we would be concerned, but finitely so. If it goes to zero, then the derivative of the value function at $x = 0$ does not affect the optimal program. So we have to examine the behavior of this expression x goes to zero. In a few cases, such as (7) above we can evaluate (25) directly and find that the limit reduces to zero. Others cases require a more indirect approach. We show two cases below.

Proposition 3.

Assume that $V(0) = B$ and that for appropriate parameters it holds that:[10]

$$V(x) \approx \tilde{V}(x) = Ax^a + B \tag{26}$$

Then (25) reduces to zero.

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[10] Obviously, if we already have the explicit expression for $\mu(x)$ as in (7) we do not need the approximation.

Proof

We have that

$$\lim_{\alpha\to 1}\mu\Big(x\big(\tau^-\big)\big(1-\alpha\big)\Big)\big(1-\alpha\big) = \lim_{y\to 0}\frac{\mu\Big(x\big(\tau^-\big)y\Big)}{\frac{1}{y}} \approx \lim_{y\to 0}\frac{Aax\big(\tau^-\big)^{a-1}y^{a-1}}{y^{-1}} = \lim_{y\to 0}Aax\big(\tau^-\big)^{a-1}y^{a} = 0 \qquad (27)$$

■

With appropriate parameters, this approximation is valid for small x for many functions satisfying $V(x) = B$ and $\lim_{x\to 0}V'\big(x\big) = \infty$. Thus Equation (22) can be rewritten

$$\dot{\mu}_r = \rho\mu_r - F_x'\big(x,u^*\big) - \mu_r f_x'\big(x,u^*\big) - \lambda\big(x,t\big)\mu_r - \lambda_x'\big(x,t\big)\Big(z - V\big(0\big)\Big) \qquad (28)$$

After rewriting (22) to (28), no trace remains of $\mu(x)$ in the optimality conditions and it does not affect the solution.

In some cases we do not need an approximation. Assume e.g. that the value function is logarithmic, that is $V(x) = A + B\times\ln(x)$. This will typically be the case when instantaneous utility is logarithmic. Then we can calculate that:

$$\mu\Big(\big(1-\alpha\big)x\big(\tau^-\big)\Big)\big(1-\alpha\big) = \frac{B\big(1-\alpha\big)}{\big(1-\alpha\big)x\big(\tau^-\big)} = \frac{B}{x\big(\tau^-\big)} > 0 \qquad (29)$$

Thus with a logarithmic value function, the optimal solution to the pre-catastrophe problem depends to some extent on the post-disaster shadow price, but in a limited manner unless *x prior* to a shock happens to be very close to zero. Note that this implies that even if the total willingness to pay to avoid such a situation is infinity, the effect of the infinite shadow price is bounded.

From (24) one can also see that that if the hazard rate is endogenous, then the post-catastrophe value function $V(0)$ plays a role. E.g. if $V(0)$ is very negative and $\lambda_x'\left(x,t\right) > 0$, then $\dot{\mu}_r$ will change very quickly so that x is driven to regions where risk is reduced or the consequences of a catastrophe are less dramatic. But in this case it is not the shadow price after a catastrophe that drives the change, it is the difference in the levels of utility before and after a catastrophe that is important. Also note that if the hazard rate is exogenous so that $\lambda_x'\left(x,t\right) = 0$, then the magnitude of $V(0)$ does not matter either.

4. Discussion

We have shown that value functions with infinite derivatives for critical values turn up endogenously in problems where none of the functions specified in the problem shares this property. The explanation is that regenerative processes exhibit infinite derivatives if the intrinsic growth rate is larger than the discount rate. However, the infinite derivatives property does not exclude that it may be optimal to end in such a system. Only that it will never be optimal to do so as an incremental process. However, if the system is subject to catastrophic shocks, it may very well be optimal that we accept a positive probability of ending up permanently in a state with a value function with infinite derivative. In a fishery example we even show that it is optimal to let the stock become extinct with probability 1 even though in the absence of a risk of catastrophe, it will never be optimal to let the stock become extinct, as long as the intrinsic growth rate is larger than the discount rate. The fishery example is linear-quadratic, and it should be clear that if value functions with infinite derivatives at critical points can turn up even in this simple model, it can also turn up in many other applications where the functional forms do not have derivatives that go to infinity at any point.

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Appendix

The solution to (16).

The solution in (17) can be found using the following calculations:

Integrating (16) over $\left[x, x^*\right)$ gives:

$$\begin{aligned}
\int_x^{x^*} \frac{1}{\mu(x)} \frac{d\mu}{dy} dy &= \int_x^{x^*} \frac{\rho}{g(y)} dy - \int_x^{x^*} \frac{g'(y)}{g(y)} dy \\
-\int_{\mu(x^*)}^{\mu(x)} \frac{1}{\mu} d\mu &= \int_x^{x^*} \frac{\rho}{g(y)} dy - \left(\ln\left(g\left(x^*\right)\right) - \ln\left(g(x)\right)\right) \\
\ln\left(\frac{\mu(x)}{\mu\left(x^*\right)}\right) &= -\int_x^{x^*} \frac{\rho}{g(y)} dy + \ln\left(\frac{g\left(x^*\right)}{g(x)}\right) \\
\frac{\mu(x)}{\mu\left(x^*\right)} &= \frac{g\left(x^*\right)}{g(x)} \exp\left(-\int_x^{x^*} \frac{\rho}{g(y)} dy\right)
\end{aligned}$$

Inserting for $\mu\left(x^*\right) = p$ and rearranging gives the expression for $\mu(x)$ in (17):

$$\mu(x) = \frac{p g\left(x^*\right)}{g(x)} \exp\left(-\int_x^{x^*} \frac{\rho}{g(y)} dy\right)$$

Calculating the expression in (17).

Applying L'Hôpital's rule to (16) yields

$$\begin{aligned}
\lim_{x\to 0}\mu\left(x\right) &= pg\left(x*\right)\lim_{x\to 0}\frac{\dfrac{d}{dx}\left(\exp\left(-\displaystyle\int_x^{x^*}\frac{\rho}{g\left(y\right)}dy\right)\right)}{g'\left(x\right)} \\
&= pg\left(x*\right)\lim_{x\to 0}\frac{\exp\left(-\displaystyle\int_x^{x^*}\frac{\rho}{g\left(y\right)}dy\right)\dfrac{\rho}{g\left(x\right)}}{g'\left(x\right)} \\
&= pg\left(x*\right)\lim_{x\to 0}\frac{\rho}{g'\left(x\right)}\frac{\exp\left(-\displaystyle\int_x^{x^*}\frac{\rho}{g\left(y\right)}dy\right)}{g\left(x\right)} \\
&= \frac{\rho}{g'\left(0\right)}\lim_{x\to 0}\frac{pg\left(x*\right)}{g\left(x\right)}\exp\left(-\int_x^{x^*}\frac{\rho}{g\left(y\right)}dy\right)
\end{aligned}$$

The last line implies that:

$$\lim_{x\to 0}\mu\left(x\right) = \frac{\rho}{g'\left(0\right)}\lim_{x\to 0}\mu\left(x\right)$$