Technological Change: A Burden or a Chance*

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Abstract

In this paper we consider a firm that has to deal with technological change facing a declining profit stream for its established product. The firm has the following options to choose from: it can either exit the industry or invest in a new technology with which it can produce an innovative product. Furthermore, it has the possibility to temporarily suspend operations before taking an irreversible decision. In case the firm decides to launch the new product we analyze the firm's optimal capacity choice opposite to a scenario where the firm has infinity capacity at its disposal. We find that depending on this assumption the investment threshold is monotonic or non-monotonic as a function of uncertainty. It is monotonic under capacity choice and non-monotonic under full flexibility. Contrary to standard real options theory we find that the effect of uncertainty on the exit threshold is non-monotonic when taking into account the capacity choice decision. Furthermore, we conduct an analysis of the effect of capacity size of the old market on the investment decision in a new product as well as the exit decision. We find that in case the firm invests out of suspension the chosen capacity level stays constant for different sizes of the old market's production capacity. Regarding the firm's suspension option, we conclude that the firm suspends operation only when uncertainty is high and innovative product market is considered very attractive.

1 Introduction

The photography industry underwent a disruptive change in technology during the 1990s when the traditional film was replaced by digital photography (see e.g. The Economist January 14th 2012). In particular Kodak was largely affected : by 1976 Kodak accounted for 90% of film and 85% of camera sales in America. Hence it was a near-monopoly in America. Kodak's revenues were nearly 16 billion in 1996 but the prediction is that it will decrease to 6.2 billion in 2011.

Kodak tried to get (squeeze) as much money out of the film business as possible and it prepared for the switch to digital film. The result was that Kodak did eventually build a profitable business out of digital cameras but it lasted only a few years before camera phones overtook it.

According to Mr Komori, the former CEO of Fujifilm of 2000-2003, Kodak aimed to be a digital company, but that is a small business and not enough to support a big company. For Kodak it was like seeing a tsunami coming and there is nothing you can do about it, according to Mr. Christensen in The Economist (January 14th 2012).

This paper focuses on industries that have to deal with technological change. The above example showed that this can be a burden. However there are enough industries where technological change brings fruitful times in terms of profits. One example is the video game industry, where innovation plays a big role. The publishers, Activision, saw their worldwide sales increase with $650m in the first five days, when the new video game "Call of Duty: Black Ops" replaced its predecessor, Call of Duty: Modern Warfare 2, in November 2010 (The Economist, December 10th 2011). Another example is the IPhone launched by Apple that was described by Time Magazine as "the invention of the year 2007". In 2011 net income was $7.31bn in the three months to 25 June, 125% higher than a year earlier and a record quarterly profit for the firm. Revenue was $28.6bn, also a quarterly record.

We study the problem of a price setting firm that produces with a current technology for which it faces a declining sales volume. The firm has three options: it can either exit this industry, invest in a new technology with which it can produce an innovative product or suspend production for a certain amount of time. The firm is a monopolist in a market characterized by uncertain demand. Demand is driven by a demand intercept assumed to follow a geometric Brownian motion. We distinguish between two scenarios in the sense that the resulting new market can be booming or ends up to be smaller than the old market used to be. We assume that the technology necessary to bring this innovative product on the market is already available to the company. This technology was either provided by other firms or was invented by the R&D laboratory of the firm itself.

The question we study is when and if it is optimal to launch this product. In case the firm decides to launch the new product we also analyze the optimal capacity choice with which the firm decides to enter the innovative product market. Since the new technology is already present, our model does not analyze the innovation process concerning the invention of the technology. Besides, adopting the new technology the firm has the option to exit the market at any point in time, i.e. the firm can decide to exit the market of the old product because it considers the potential of the new product market not profitable enough to invest and thus, decides to exit before launching the new product. The exit option is conserved beyond the time of the potential investment in the new product. Thus, the firm can also exit the market of the new product irrevocably at any time.

Before taking the irreversible decision to exit or invest, the firm furthermore has the possibility to temporarily suspend production. This suspension option results in three possible scenarios regarding the market of the current technology. The firm might never suspend production because it is optimal to exit the market before already. In the second scenario operation might not be profitable anymore at all considering the current product market and the firm has to decide whether to exit or invest in the innovative market out of suspension mode. In the third scenario the firm once suspended, decides to exit the market if demand continues to fall or resume production if demand increases again in the future.

We find that the firm uses the option to suspend operation only when uncertainty is high, and the parameters that make the second market attractive are high. For the specific case that the firm invests out of suspension we find that the chosen capacity level stays constant in different values of the current market capacity size as well as in the scaling parameter of the demand intercept γ. The reason for this is that the firm gives up the same revenue stream independent of γ when investing in the new product since it does not produce before investment. In fact, we find that the firm invests such that also after the investment the revenue stream is the same independent of γ. Therefore, the capacity choice for the new market is insensitive to γ.

We introduce our model in Section 3 and derive the result that the optimal policy of the considered stopping problem exits and is unique. In Section 4 we specify a model assuming that the firm has infinite capacity available. This means we do not consider the capacity decision. In this case the firm is fully flexible regarding production and can downscale, increase or suspend production at any point in time. We use this model as a benchmark in the analysis of our main model presented in Section 3. Analyzing the investment timing decision of the benchmark model we find that the effect of uncertainty on the investment threshold is contrary to the standard real options result non-monotonic. The investment threshold increases in uncertainty for low values while it decreases for high values of uncertainty due to the existence of the exit option. A similar result was obtained by Kwon (2010) who analyzes a one-time opportunity to invest in improving a current technology with declining profit stream. He finds that, if the technology boost is sufficiently large, then the investment threshold decreases in demand uncertainty. However, we show that in case the investment decision does not only involve timing but also the choice of the optimal capacity size, the investment threshold is again monotonically increasing in uncertainty. Regarding the exit decision, the standard real options result says that the exit threshold is decreasing with uncertainty (Dixit (1989), Kwon (2010)) since the value of waiting increases. We find that this result, however, does not hold anymore when the investment decision involves timing and capacity choice. We show that for our framework the effect of uncertainty on the exit threshold is non-monotonic where the increasing part is caused by the effect of capacity choice.

This paper is organized as follows. We review related literature in Section 2. Our model is presented in Section 3 followed by a simplified version that serves as benchmark in the analysis. The comparative statics analysis of the optimal policies for both models is conducted in Section. Our main results are present in Section 6 and we conclude in Section 7.

2 Related Literature

WORK IN PROGRESS

3 Model - Capacity, Timing

The firm currently produces an established product. The quantity, which has to be determined at each point of time, is q_1, the price is p_1 and the inverse demand function is given by

$$p_1 = \mu\theta - q_1,$$

in which the process θ_t follows the geometric Brownian motion

$$d\theta_t = \alpha_1 \theta_t dt + \sigma \theta_t dz_t,$$

with constant drift α_1 that is assumed to be negative and volatility σ.

We distinguish between two types of cost. On the one hand the firm faces a fixed cost F. On the other hand it has to incur unit production costs being equal to c. Due to the latter feature it can be optimal to temporarily suspend production, i.e. $q_1 = 0$ for some time.

The firm has the option to start producing an innovative product which requires an investment in production capacity. The capacity of the new product is denoted by K_2. The investment cost is sunk and equal to δK_2. Denoting the price and the quantity of the new product by p_2 and q_2, respectively, at the moment of the new product launch the firm's demand function changes into:

$$p_2 = \gamma\theta - q_2. \tag{1}$$

The firm produces up to full capacity, i.e. $q_2 = K_2$, except in the cases that demand falls so low that the price would turn negative. In that case the firm temporarily suspends production, i.e. $q_2 = 0$.

Because this innovative market grows faster than the old one, we assume a different speed of development. In particular the dynamics of θ now become

$$d\theta = \alpha_2 \theta dt + \sigma \theta dz. \tag{2}$$

with $\alpha_2 > \alpha_1$. In fact, we can define the stochastic process $\theta(t)$ as follows

$$d\theta(t) = \alpha(t)\theta(t)dt + \sigma\theta(t)dz,$$

with

$$\alpha(t) = \begin{cases} \alpha_1 & \text{for } t < \tau^*, \\ \alpha_2 & \text{for } t \geq \tau^*, \end{cases}$$

where this τ^* will be specified later.

The cost structure for the new product also changes after the new product launch. Whereas the fixed cost still equals F, there are no variable cost. We motivate this by observing that in the digital world the unit cost of a product is most of the time very small or negligible.

In case the firm decides to invest in the new product, the firm chooses the optimal timing as well as the optimal size of the capacity investment. It can be the case that the new market is not profitable enough for an investment to be undertaken. Since the old market is decreasing over time it can be optimal for the firm to exercise the option to exit the market. We also allow for the possibility to exit the market after the investment in the innovative product has taken place.

Therefore, the optimal stopping problem can be stated as follows:

$$\begin{aligned} \mathcal{V}(\theta_0) = \sup_{\tau_1} \mathbb{E}\Bigg[\int_0^{\tau_1} e^{-rt}\Pi_1(\theta_1(t))dt + e^{-r\tau_1}\max\{0, \\ \max_{K_2}\left(\sup_{\tau_2 \mathbf{1}_{\{\tau_2>\tau_1\}}} \mathbb{E}\left[\int_{\tau_1}^{\tau_2} e^{-r(t-\tau_1)}\Pi_2(\theta_2(t-\tau_1), K_2)dt \middle| \theta_2(0) = \theta_1(\tau_1)\right] - \delta K_2\right)\Bigg\}\Bigg|\theta_1(0) = \theta_0\Bigg] \end{aligned} \tag{3}$$

Here, τ_1 denotes the first time at which the decision maker decides to invest in product 2 or to exit the market. τ_2 denotes the time that the firm would decide to exit the market of product 2, in case it has invested in the first run.

To determine the value of investing in project 2, we first solve the subproblem that is stated at the right hand side of the maximization in equation (3). Considering a specific current value for $\theta_2(0)$ the net expected discounted profit of investing in project 2 is given by:

$$
\begin{aligned}
V_2(\theta_2(0)) &= \sup_{\tau_2} \mathbb{E}^{\theta_2(0)}\left[\int_{\tau_1}^{\tau_2} e^{-r(t-\tau_1)}\Pi_2(\theta_2(t-\tau_1))dt\right], \qquad (4)\\
&= \sup_{\tau_2} \mathbb{E}^{\theta_2(0)}\left[\int_{0}^{\tau_2-\tau_1} e^{-rt}\Pi_2(\theta_2(t))dt\right],\\
&= \sup_{\tilde{\tau}} \mathbb{E}^{\theta_2(0)}\left[\int_{0}^{\tilde{\tau}} e^{-rt}\Pi_2(\theta_2(t))dt\right],
\end{aligned}
$$

where $\mathbb{E}^{\theta}$ denotes the expectation with respect to the probability law Q^{θ} of the process $\{\theta(t); t > 0\}$ starting at $\theta(0) = \theta \in \mathbb{R}^n$. The optimal stopping problem in (4) is a standard problem. The instantaneous profits in region 2 are given by

$$\Pi_2(\theta) = p_2 q_2 - F = (\gamma\theta - K_2)K_2 - F = \gamma K_2\theta - K_2^2 - F.$$

The firm produces as long as the price is positive ($p_2 > 0$) but will suspend production in case the price turns negative. Therefore the profit is given by

$$\Pi_2(\theta) = \begin{cases} \gamma K_2\theta - K_2^2 - F & \text{for } \theta > \frac{K_2}{\gamma}, \\ -F & \text{for } \theta \le \frac{K_2}{\gamma}. \end{cases}$$

We denote the suspension threshold in market 2 by $\frac{K_2}{\gamma} = \theta_{S_2}$.

Taking into account that there is an option to exit the market, standard calculations (see, e.g. Chapter 6 in Section 2 of Dixit and Pindyck (1994)) lead to the following expression for the optimal value function V_2:

$$V_2(\theta, K_2) = \begin{cases} \frac{\gamma K_2}{r-\alpha_2}\theta - \frac{K_2^2+F}{r} + D_1\theta^{\beta_4} & \text{for } \theta > \frac{K_2}{\gamma}, \\ -\frac{F}{r} + D_2\theta^{\beta_3} + D_3\theta^{\beta_4} & \text{for } \theta \le \frac{K_2}{\gamma}, \end{cases}$$

where β_3 (β_4) is the positive (negative) root of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha_2\beta - r = 0$. Here we assume that the firm will exit out of the suspension region. However, it is also possible that the firm will exit before ever entering the suspension region (i.e. $\theta_{S_2} = \frac{K_2}{\gamma} < \theta_{E_2}$ with θ_{E_2} denoting the exit threshold). In this case the value function is given by

$$V_2(\theta, K_2) = \frac{\gamma K_2}{r-\alpha_2}\theta - \frac{K_2^2+F}{r} + G_1\theta^{\beta_4}.$$

In the following we have to distinguish those two cases. The exit threshold θ_{E2} for the two cases as well as the specific expressions of the constant parameters D_1, D_2, D_3 and G_1, can be easily derived using the fact that the value function has to be continuous and smooth in the suspension threshold and applying value matching and smooth pasting at the exit threshold. The derivations as well as the explicit expressions for the parameter values are given in Appendix A.1. The exit thresholds for the two different cases are given by

$$\theta_{E_2} = \left(\frac{\beta_4\alpha_2 - r}{\beta_4(r-\alpha_2)}\right)^{\frac{-1}{\beta_3}} F^{\frac{1}{\beta_3}} \frac{1}{\gamma} K_2^{1-\frac{2}{\beta_3}}, \qquad (5)$$

for case $\theta_{S_2} > \theta_{E_2}$ and

$$\theta_{E_2} = \frac{r-\alpha_2}{r\gamma}\frac{\beta_4}{\beta_4-1}\left(K_2 + \frac{F}{K_2}\right), \tag{6}$$

for case $\theta_{S_2} < \theta_{E_2}$.

Now we can write equation (3) as

$$\mathcal{V}(\theta_0) = \sup_{\tau_1} E\left[\int_0^{\tau_1} e^{-rt}\Pi_1(\theta_1(t))dt + e^{-r\tau_1}\max\left\{0, \max_{K_2}\left(V_2(\theta_2(0)=\theta_1(\tau_1), K_2) - \delta K_2\right)\right\}\bigg|\theta_1(0)=\theta_0\right].$$

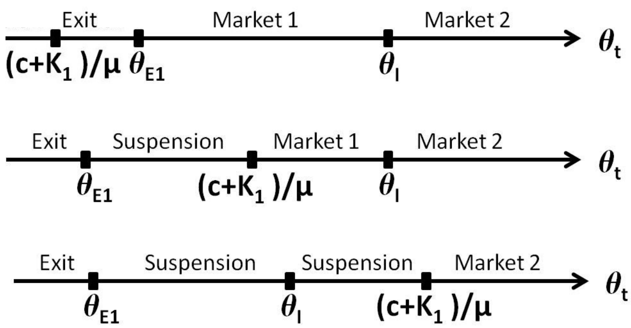
Now we consider the situation before the investment. The firm has essentially three options. The first is to invest in product 2 while in production. The second is to suspend production and in the meanwhile invest in product 2. And the third is to exit the market. Let us first determine the current instantaneous profits. The firm will suspend production when $p_1 - c = \mu\theta - K_1 - c < 0$, i.e., when $\theta < \theta_{S_1} = \frac{c+K_1}{\mu}$. The instantaneous profit equals:

$$\Pi_1(\theta) = \begin{cases} \mu K_1\theta - (cK_1 + K_1^2 + F) & \text{for } \theta > \frac{c+K_1}{\mu}, \\ -F & \text{for } \theta \le \frac{c+K_1}{\mu}. \end{cases}$$

Denoting the exit threshold by θ_{E_1} we can consider the following two cases: If $\theta_{E_1} \geq \theta_{S_1}$ then the firm will never suspend production because it already has exited. If $\theta_{E_1} < \theta_{S_1}$ there exists a θ-interval, where it is optimal for the firm to suspend production. In this region the firm has two options: either to resume production if θ has increased sufficiently or to exit the market, which will happen when θ decreases even more. In the following we will denote the invest threshold by θ_I. Figure 1 illustrates the different regions for the three options.

The following proposition states that there always exists the optimal policy for the stopping problem and specifies the optimal value function of the firm.

Figure 1: Different Cases



Proposition 1 *The optimal policy for the stopping problem of equation (3) always exists. The optimal value*

function is uniquely given by

$$\mathcal{V}(\theta) = \begin{cases} V_1(\theta) & \text{for } \theta \in D^* = (\theta_{E_1}, \theta_I), \\ \Omega(\theta) & \text{otherwise,} \end{cases}$$

where we distinguish three cases for the function $V_1(.)$. If $\theta_{E_1} \geq \frac{c+K_1}{\mu}$, the value function $V_1(.)$ is uniquely given by

$$V_1(\theta) = \frac{\mu K_1 \theta}{r - \alpha_1} - \frac{cK_1 + K_1^2 + F}{r} + A_1 \theta^{\beta_1} + A_2 \theta^{\beta_2}.$$

For case $\theta_{E_1} < \frac{c+K_1}{\mu}$ the value function is equal to

$$V_1(\theta) = \begin{cases} \frac{\mu K_1 \theta}{r-\alpha_1} - \frac{cK_1+K_1^2+F}{r} + B_1\theta^{\beta_1} + B_2\theta^{\beta_2} & \text{for } \theta \geq \frac{c+K_1}{\mu}, \\ -\frac{F}{r} + B_3\theta^{\beta_1} + B_4\theta^{\beta_2} & \text{for } \theta < \frac{c+K_1}{\mu}, \end{cases}$$

if $\theta_I \geq \frac{c+K_1}{\mu}$ and equal to

$$V_1(\theta) = -\frac{F}{r} + C_1\theta^{\beta_1} + C_2\theta^{\beta_2},$$

in case $\theta_I < \frac{c+K_1}{\mu}$.

The value of the firm in the stopping region is equal to $\Omega(\theta) = \max\{0, V_2(\theta_2(0) = \theta, K_2)\}$. The optimal continuation region is $D^* = (\theta_{E_1}, \theta_I)$. It is optimal to exit the market when $\theta < \theta_{E_1}$ and invest in the new product when $\theta > \theta_I$. Otherwise, it is optimal to continue operations.

In case the firm decides to invest in the new product instead of exiting, it does choose the timing as well as the size of this investment. Deriving this optimal investment decision we follow **?**. First, we ignore the timing decision for a moment and just take into account the derivation of the optimal capacity. Second, we derive the optimal timing of the investment or exit decision, respectively, taking the optimal capacity size for given demand intercept θ in account. It is important to note that we wish to compute $\max_{K_2 \in [0, \gamma\theta_I]} \{V_2(\theta_I, K_2) - \delta K_2\}$. So if this maximum is not on the boundary we can just compute the zero of $\frac{\partial(V_2(\theta_I, K_2) - \delta K_2)}{\partial K_2}$ and then check that $\frac{\partial^2(V_2(\theta_I, K_2) - \delta K_2)}{\partial^2 K_2} < 0$. Otherwise the maximum will be at the boundary.

Proposition 2 gives the optimal capacity decision as well as the equations that implicity give the exit as well as the investment decision for the different cases. To compute the optimal K_2 we write the expression of the derivate, for the case that the maximum is not on the boundary. Note that we have to distinguish 6 cases.

Proposition 2 *The exit and investment thresholds for the six different cases, respectively, are given implicitly by the following equations.*
For the purpose of readability let's consider the following constants:

$$a_1 = \frac{\mu K_1}{r-\alpha_1}; a_2 = \frac{cK_1 + K_1^2 + F}{r}; a_3 = \frac{\gamma}{r-\alpha_2}; a_4 = \frac{r-\alpha_2}{\gamma r}\frac{\beta_4}{\beta_4 - 1}$$
$$a_5 = \frac{\gamma^{\beta_4}(r-\beta_3\alpha_2)}{(\beta_3-\beta_4)r(r-\alpha_2)}; a_6 = \gamma^{\beta_4}\left(\frac{\beta_4\alpha_2 - r}{\beta_4(r-\alpha_2)}\frac{1}{F}\right)^{\frac{\beta_4}{\beta_3}}\frac{F}{r}\frac{\beta_3}{\beta_3-\beta_4}$$

For the first three cases it holds that $\theta_{E_2} \le \theta_{S_2}$. Therefore if K_2 is not on the boundary, K_2 is implicitly given by the following equation:

$$a_3\theta_I - \frac{2K_2}{r} + \left(a_5(2-\beta_4)K_2^{1-\beta_4} + a_6\frac{(-\beta_4)(\beta_3-2)}{\beta_3}K_2^{-\beta_4\left(1-\frac{2}{\beta_3}\right)-1}\right)\theta_I^{\beta_4} = \delta \quad (7)$$

The threshold for those three cases, is implicitly given by the solution of three subsequent equations, respectively.

1. ***Case 1:*** *In case of $\theta_{E_1} > \theta_{S_1}$ and $\theta_{E_2} \le \theta_{S_2}$ it holds that*

$$(\beta_1 - 1)\left[a_1\theta_{E_1}^{1-\beta_2} + (a_3K_2 - a_1)\theta_I^{1-\beta_2}\right] - \beta_1\left[a_2\theta_{E_1}^{-\beta_2} - \left(a_2 - \frac{K_2^2+F}{r} - \delta K_2\right)\theta_I^{-\beta_2}\right] +$$
$$(\beta_1-\beta_4)\left[a_5K_2^{2-\beta_4} + a_6K_2^{-\beta_4\left(1-\frac{2}{\beta_3}\right)}\right]\theta_I^{\beta_4-\beta_2} = 0 \quad (8)$$
$$(\beta_2 - 1)\left[a_1\theta_{E_1}^{1-\beta_1} + (a_3K_2 - a_1)\theta_I^{1-\beta_1}\right] - \beta_2\left[a_2\theta_{E_1}^{-\beta_1} - \left(a_2 - \frac{K_2^2+F}{r} - \delta K_2\right)\theta_I^{-\beta_1}\right] +$$
$$(\beta_2-\beta_4)\left[a_5K_2^{2-\beta_4} + a_6K_2^{-\beta_4\left(1-\frac{2}{\beta_3}\right)}\right]\theta_I^{\beta_4-\beta_1} = 0 \quad (9)$$

2. ***Case 2:*** *In case of $\theta_{E_1} < \theta_{S_1} < \theta_I$ and $\theta_{E_2} \le \theta_{S_2}$ it holds that*

$$\theta_{E_1} = \Bigg(\Bigg((\beta_1-1)\left[a_1\left(\frac{c+K_1}{\mu}\right)^{1-\beta_2} + (a_3K_2-a_1)\theta_I^{1-\beta_2}\right] -$$
$$\beta_1\left[\frac{cK_1+K_1^2}{r}\left(\frac{c+K_1}{\mu}\right)^{-\beta_2} - \left(a_2 - \frac{K_2^2+F}{r} - \delta K_2\right)\theta_I^{-\beta_2}\right] +$$
$$(\beta_1-\beta_4)\left[a_5K_2^{2-\beta_4} + a_6K_2^{-\beta_4\left(1-\frac{2}{\beta_3}\right)}\right]\theta_I^{\beta_4-\beta_2}\Bigg)\frac{1}{\beta_1}\frac{r}{F}\Bigg)^{-\frac{1}{\beta_2}}; \quad (10)$$
$$(\beta_2-1)\left[a_1\left(\frac{c+K_1}{\mu}\right)^{1-\beta_1} + (a_3K_2-a_1)\theta_I^{1-\beta_1}\right] -$$
$$\beta_2\left[\frac{cK_1+K_1^2}{r}\left(\frac{c+K_1}{\mu}\right)^{-\beta_1} - \left(a_2 - \frac{K_2^2+F}{r} - \delta K_2\right)\theta_I^{-\beta_1}\right] +$$
$$(\beta_2-\beta_4)\left[a_5K_2^{2-\beta_4} + a_6K_2^{-\beta_4\left(1-\frac{2}{\beta_3}\right)}\right]\theta_I^{\beta_4-\beta_1} - \beta_2\theta_{E_1}^{-\beta_1}\frac{F}{r} = 0 \quad (11)$$

3. ***Case 3:*** *In case of* $\theta_{E_1} < \theta_I < \theta_{S_1}$ *and* $\theta_{E_2} \leq \theta_{S_2}$ *it holds that*

$$\begin{aligned} \theta_{E_1} = \Bigg(\Bigg((\beta_1 - 1)a_3 K_2 \theta_I^{1-\beta_2} - \beta_1 \left(\frac{K_2^2}{r} + \delta K_2\right)\theta_I^{-\beta_2} \quad + \\ (\beta_1 - \beta_4)\left[a_5 K_2^{2-\beta_4} + a_6 K_2^{-\beta_4\left(1-\frac{2}{\beta_3}\right)}\right]\theta_I^{\beta_4-\beta_2}\Bigg)\frac{1}{\beta_1}\frac{r}{F}\Bigg)^{-\frac{1}{\beta_2}}; \end{aligned} \tag{12}$$

$$\begin{aligned} (\beta_2 - 1)a_3 K_2 \theta_I^{1-\beta_1} - \beta_2\left(\frac{K_2^2}{r} + \delta K_2\right)\theta_I^{-\beta_1} \quad + \\ (\beta_2 - \beta_4)\left[a_5 K_2^{2-\beta_4} + a_6 K_2^{-\beta_4\left(1-\frac{2}{\beta_3}\right)}\right]\theta_I^{\beta_4-\beta_1} - \beta_2 \theta_{E_1}^{-\beta_1}\frac{F}{r} \quad = \quad 0 \end{aligned} \tag{13}$$

Regarding the remaining three cases it holds that $\theta_{E_2} > \theta_{S_2}$ *and therefore if* K_2 *is not on the boundary, it is implicitly given by the solution of the following equation:*

$$a_3\theta_I - \frac{2K_2}{r} - \left(\frac{\beta_4 F + K_2^2(2-\beta_4)}{K_2 r(\beta_4 - 1)}\left(a_4\left(K_2 + \frac{F}{K_2}\right)\right)^{-\beta_4}\right)\theta_I^{\beta_4} = \delta \tag{14}$$

The threshold for those three cases, is implicitly given by the solution of three subsequent equations, respectively:

4. ***Case 4:*** *In case of* $\theta_{E_1} > \theta_{S_1}$ *and* $\theta_{E_2} > \theta_{S_2}$ *it holds that*

$$\begin{aligned} (\beta_1 - 1)\left[a_1\theta_{E_1}^{1-\beta_2} + (a_3K_2 - a_1)\theta_I^{1-\beta_2}\right] - \beta_1\left[a_2\theta_{E_1}^{-\beta_2} - \left(a_2 - \frac{K_2^2 + F}{r} - \delta K_2\right)\theta_I^{-\beta_2}\right] \quad - \\ (\beta_1 - \beta_4)\left[\frac{\gamma K_2}{r - \alpha_2}\frac{1}{\beta_4}\left(a_4\left(K_2 + \frac{F}{K_2}\right)\right)^{1-\beta_4}\right]\theta_I^{\beta_4-\beta_2} \quad = \quad 0 \end{aligned} \tag{15}$$

$$\begin{aligned} (\beta_2 - 1)\left[a_1\theta_{E_1}^{1-\beta_1} + (a_3K_2 - a_1)\theta_I^{1-\beta_1}\right] - \beta_2\left[a_2\theta_{E_1}^{-\beta_1} - \left(a_2 - \frac{K_2^2 + F}{r} - \delta K_2\right)\theta_I^{-\beta_1}\right] \quad - \\ (\beta_2 - \beta_4)\left[\frac{\gamma K_2}{r - \alpha_2}\frac{1}{\beta_4}\left(a_4\left(K_2 + \frac{F}{K_2}\right)\right)^{1-\beta_4}\right]\theta_I^{\beta_4-\beta_1} \quad = \quad 0 \end{aligned} \tag{16}$$

5. ***Case 5:*** *In case of* $\theta_{E_1} < \theta_{S_1} < \theta_I$ *and* $\theta_{E_2} > \theta_{S_2}$ *it holds that*

$$\begin{aligned}
\theta_{E_1} = \Bigg(\Bigg(&(\beta_1 - 1)\left[a_1\left(\frac{c+K_1}{\mu}\right)^{1-\beta_2} + (a_3K_2 - a_1)\theta_I^{1-\beta_2}\right] \quad - \\
&\beta_1\left[\frac{cK_1 + K_1^2}{r}\left(\frac{c+K_1}{\mu}\right)^{-\beta_2} - \left(a_2 - \frac{K_2^2+F}{r} - \delta K_2\right)\theta_I^{-\beta_2}\right] \quad - \\
&(\beta_1 - \beta_4)\left[\frac{\gamma K_2}{r-\alpha_2}\frac{1}{\beta_4}\left(a_4\left(K_2 + \frac{F}{K_2}\right)\right)^{1-\beta_4}\right]\theta_I^{\beta_4-\beta_2}\Bigg)\frac{1}{\beta_1}\frac{r}{F}\Bigg)^{-\frac{1}{\beta_2}}; \qquad (17)\\
&(\beta_2 - 1)\left[a_1\left(\frac{c+K_1}{\mu}\right)^{1-\beta_1} + (a_3K_2 - a_1)\theta_I^{1-\beta_1}\right] \quad - \\
&\beta_2\left[\frac{cK_1 + K_1^2}{r}\left(\frac{c+K_1}{\mu}\right)^{-\beta_1} - \left(a_2 - \frac{K_2^2+F}{r} - \delta K_2\right)\theta_I^{-\beta_1}\right] \quad - \\
&(\beta_2 - \beta_4)\left[\frac{\gamma K_2}{r-\alpha_2}\frac{1}{\beta_4}\left(\left(K_2 + \frac{F}{K_2}\right)\frac{r-\alpha_2}{\gamma r}\frac{\beta_4}{\beta_4 - 1}\right)^{1-\beta_4}\right]\theta_I^{\beta_4-\beta_1} - \beta_2\theta_{E_1}^{-\beta_1}\frac{F}{r} \quad = \quad 0 \qquad (18)
\end{aligned}$$

6. ***Case 6:*** *In case of* $\theta_{E_1} < \theta_I < \theta_{S_1}$ *and* $\theta_{E_2} > \theta_{S_2}$ *it holds that*

$$\begin{aligned}
\theta_{E_1} = \Bigg(\Bigg(&(\beta_1 - 1)a_3K_2\theta_I^{1-\beta_2} - \beta_1\left(\frac{K_2^2}{r} + \delta K_2\right)\theta_I^{-\beta_2} \quad + \\
&(\beta_1 - \beta_4)\left[\frac{\gamma K_2}{r-\alpha_2}\frac{1}{(-\beta_4)}\left(\left(K_2 + \frac{F}{K_2}\right)\frac{r-\alpha_2}{\gamma r}\frac{\beta_4}{\beta_4 - 1}\right)^{1-\beta_4}\right]\theta_I^{\beta_4-\beta_2}\Bigg)\frac{1}{\beta_1}\frac{r}{F}\Bigg)^{-\frac{1}{\beta_2}}; \qquad (19)\\
&(\beta_2 - 1)a_3K_2\theta_I^{1-\beta_1} - \beta_2\left(\frac{K_2^2}{r} + \delta K_2\right)\theta_I^{-\beta_1} \quad - \\
&(\beta_2 - \beta_4)\left[\frac{\gamma K_2}{r-\alpha_2}\frac{1}{\beta_4}\left(\left(K_2 + \frac{F}{K_2}\right)\frac{r-\alpha_2}{\gamma r}\frac{\beta_4}{\beta_4 - 1}\right)^{1-\beta_4}\right]\theta_I^{\beta_4-\beta_1} - \beta_2\theta_{E_1}^{-\beta_1}\frac{F}{r} \quad = \quad 0 \qquad (20)
\end{aligned}$$

4 Analysis of Benchmark Model

In this section we present a model that will serve as a benchmark for the analysis of the model introduced in Section 3. We now assume that the firm has infinite capacity. Therefore, including a parameter K_2 is not necessary. In order to start producing an innovative product, the firm has to pay sunk cost I. All other assumptions remain unchanged. Therefore the optimal stopping problem does not differ much from the previous case:

$$\begin{aligned}
\mathcal{V}(\theta_0) = \sup_{\tau_1}\mathbb{E}\Bigg[&\int_0^{\tau_1} e^{-rt}\Pi_1(\theta_1(t))dt + e^{-r\tau_1}\max\{0, \\
&\sup_{\tau_2 \mathbf{1}_{\{\tau_2 > \tau_1\}}}\mathbb{E}\left[\int_{\tau_1}^{\tau_2} e^{-r(t-\tau_1)}\Pi_2(\theta_2(t-\tau_1))dt \,\Big|\, \theta_2(0) = \theta_1(\tau_1)\right] - I\Bigg\}\,\Bigg|\,\theta_1(0) = \theta_0\Bigg], \qquad (21)
\end{aligned}$$

Here, τ_1 denotes the first time at which the decision maker decides to invest in product 2 or exit the market. τ_2 denotes the time that the firm would decide to exit the market of product 2, in case it has invested in the first run.

As before, V_2 is given by:

$$V_2(\theta_2(0)) = \sup_{\tilde{\tau}} \mathbb{E}^{\theta_2(0)} \left[\int_0^{\tilde{\tau}} e^{-rt} \Pi_2(\theta_2(t)) dt \right].$$

The instantaneous profits in region 2 are given by

$$\Pi_2 = p_2 q_2 - F = (\gamma\theta - q_2) q_2 - F.$$

In this model, we compute the optimal quantity to produce. That quantity equals $q_2 = \frac{\gamma\theta}{2}$, which results in a profit of

$$\Pi_2(\theta) = \frac{\gamma^2\theta^2}{4} - F.$$

Taking into account that there is an option to exit the market, standard calculations, given in Appendix A.3, lead to the following expression for the optimal value function V_2:

$$V_2(\theta) = \frac{\gamma^2\theta^2}{4(r - 2\alpha_2 - \sigma^2)} - \frac{F}{r} + A\theta^{\beta_4}.$$

where β_4 is again the negative root of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha_2\beta - r = 0$. After similar computations as in the previous model, we obtain the following expression for θ_{E_2}:

$$\theta_{E_2} = \sqrt{\frac{F}{\gamma^2 r} 4(r - 2\alpha_2 - \sigma^2) \left(\frac{\beta_4}{\beta_4 - 2} \right)}. \tag{22}$$

Now we consider the situation before the investment. Let us first determine the current instantaneous profits. Since there is no restriction in the production quantity, maximizing the profit function w.r.t to the optimal output quantity, we derive that:

$$q_1 = \frac{\mu\theta - c}{2}.$$

From this expression we see that the firm will suspend production, provided that it has not exited already, whenever θ is below $\frac{c}{\mu}$. The instantaneous profit, therefore, equals:

$$\Pi_1(\theta) = \begin{cases} \left(\frac{\mu\theta - c}{2}\right)^2 - F & \text{for } \theta > \frac{c}{\mu}, \\ -F & \text{for } \theta \leq \frac{c}{\mu}. \end{cases}$$

Denoting the exit threshold by θ_{E_1} we can consider the following two cases: If $\theta_{E_1} \geq \frac{c}{\mu}$ then the firm will never suspend production because it already has exited. If $\theta_{E_1} < \frac{c}{\mu}$ there exists a θ-interval, where it is optimal for the firm to suspend production. In this region the firm has two options: either to resume production if θ has increased sufficiently or to exit the market, which will happen when θ decreases even more. $\theta_S = \frac{c}{\mu}$ denotes the suspension threshold. The following proposition states that there always exists the optimal policy for the stopping problem and specifies the optimal value function of the firm.

Proposition 3 *The optimal policy for the stopping problem of equation (21) always exists. The optimal value function is uniquely given by*

$$\mathcal{V}(\theta) = \begin{cases} V_1(\theta) & \text{for } \theta \in D^* = (\theta_{E_1}, \theta_I), \\ \Omega(\theta) & \text{otherwise,} \end{cases}$$

where we distinguish two cases for the function $V_1(.)$. If $\theta_{E_1} \geq \frac{c}{\mu}$ then the function is equal to

$$V_1(\theta) = \frac{\mu^2\theta^2}{4(r - 2\alpha_1 - \sigma^2)} - \frac{c\mu\theta}{2(r - \alpha_1)} + \frac{c^2}{4r} - \frac{K}{r} + A_1\theta^{\beta_1} + A_2\theta^{\beta_2},$$

while for case $\theta_{E_1} < \frac{c}{\mu}$ the function is equal to

$$V_1(\theta) = \begin{cases} \frac{\mu^2\theta^2}{4(r-2\alpha_1-\sigma^2)} - \frac{c\mu\theta}{2(r-\alpha_1)} + \frac{c^2}{4r} - \frac{K}{r} + B_1\theta^{\beta_1} + B_2\theta^{\beta_2} & \text{for } \theta \geq \frac{c}{\mu}, \\ -\frac{K}{r} + B_3\theta^{\beta_1} + B_4\theta^{\beta_2} & \text{for } \theta < \frac{c}{\mu}, \end{cases}$$

if $\theta_I > \frac{c}{\mu}$ and equal to

$$V_1(\theta) = -\frac{K}{r} + C_1\theta^{\beta_1} + C_2\theta^{\beta_2},$$

if $\theta_I < \frac{c}{\mu}$. The value of the firm in the stopping region is equal to $\Omega(\theta) = \max\left(0, V_2(\theta_2(0) = \theta)\right)$. The optimal continuation region is $D^* = (\theta_{E_1}, \theta_I)$. It is optimal to exit the market when $\theta < \theta_{E_1}$ and invest in the new product when $\theta > \theta_I$. Otherwise, it is optimal to continue operations.

In order to derive the two thresholds θ_I and θ_{E_1} we apply the value matching and smooth pasting conditions which leads to the following systems of equations, stated in Proposition 4, that implicitly define the thresholds for the different cases.

Proposition 4 *Considering the following constants,*

$$b_1 = \frac{\mu^2}{4(r - 2\alpha_1 - \sigma^2)}; b_2 = \frac{c\mu}{2(r - \alpha_1)}; b_3 = \frac{c^2}{4r} - \frac{F}{r}; b_4 = \frac{\gamma^2}{4(r - 2\alpha_2 - \sigma^2)}; b_5 = \frac{F}{r},$$

the investment and exit threshold for the three cases, respectively, are implicitly given by the following equations.

If $\theta_{E_1} > \frac{c}{\mu}$ the thresholds θ_I and θ_{E_1} are implicitly given by

$$\begin{aligned} (\beta_2 - 2)\left[b_1\theta_{E_1}^{2-\beta_1} - (b_1 - b_4)\theta_I^{2-\beta_1}\right] &- b_2(\beta_2 - 1)\left[\theta_{E_1}^{1-\beta_1} - \theta_I^{1-\beta_1}\right] + b_3\beta_2\left[\theta_{E_1}^{-\beta_1} - \theta_I^{-\beta_1}\right] \\ &- \beta_2(b_5 + I)\theta_I^{-\beta_1} + A\theta_I^{\beta_4-\beta_1}(\beta_2 - \beta_4) = 0, \end{aligned} \quad (23)$$

$$\begin{aligned} (\beta_1 - 2)\left[b_1\theta_{E_1}^{2-\beta_2} - (b_1 - b_4)\theta_I^{2-\beta_2}\right] &- b_2(\beta_1 - 1)\left[\theta_{E_1}^{1-\beta_2} - \theta_I^{1-\beta_2}\right] + b_3\beta_1\left[\theta_{E_1}^{-\beta_2} - \theta_I^{-\beta_2}\right] \\ &- \beta_1(b_5 + I)\theta_I^{-\beta_2} + A\theta_I^{\beta_4-\beta_2}(\beta_1 - \beta_4) = 0. \end{aligned} \quad (24)$$

For the case that $\theta_{E_1} < \frac{c}{\mu} < \theta_I$ it holds that the thresholds are defined by the following equations system.

$$\begin{aligned}
\theta_{E_1} = \Bigg(\frac{1}{b_5\beta_2}\Bigg((\beta_2-2)\left[b_1\left(\frac{c}{\mu}\right)^{2-\beta_1} - (b_1-b_4)\theta_I^{2-\beta_1}\right] &\quad - \\
b_2\,(\beta_2-1)\left[\left(\frac{c}{\mu}\right)^{1-\beta_1} - \theta_I^{1-\beta_1}\right] + (b_3+b_5)\beta_2\left[\left(\frac{c}{\mu}\right)^{-\beta_1} - \theta_I^{-\beta_1}\right] &\quad - \\
\beta_2 I\theta_I^{-\beta_1} + A\theta_I^{\beta_4-\beta_1}\,(\beta_2-\beta_4)\Bigg)\Bigg)^{-\frac{1}{\beta_1}}, & \qquad (25) \\
(\beta_1-2)\left[b_1\left(\frac{c}{\mu}\right)^{2-\beta_2} - (b_1-b_4)\theta_I^{2-\beta_2}\right] - b_2\,(\beta_1-1)\left[\left(\frac{c}{\mu}\right)^{1-\beta_2} - \theta_I^{1-\beta_2}\right] &\quad + \\
(b_3+b_5)\beta_1\left[\left(\frac{c}{\mu}\right)^{-\beta_2} - \theta_I^{-\beta_2}\right] &\quad - \\
\beta_1 I\theta_I^{-\beta_2} + A\theta_I^{\beta_4-\beta_2}\,(\beta_1-\beta_4) - b_5\beta_1\theta_{E_1}^{-\beta_2} &= 0. \qquad (26)
\end{aligned}$$

And for the case that $\theta_{E_1} < \theta_I < \frac{c}{\mu}$ the following equations implicitly define the thresholds.

$$\theta_{E_1} = \left(\frac{1}{b_5\beta_2}\left(b_4\,(\beta_2-2)\,\theta_I^{2-\beta_1} - I\beta_2\theta_I^{-\beta_1} + A\,(\beta_2-\beta_4)\,\theta_I^{\beta_4-\beta_1}\right)\right)^{-\frac{1}{\beta_1}}, \qquad (27)$$

$$b_4\,(\beta_1-2)\,\theta_I^{2-\beta_2} - I\beta_1\theta_I^{-\beta_2} + A\,(\beta_1-\beta_4)\,\theta_I^{\beta_4-\beta_2} - b_5\beta_1\theta_{E_1}^{-\beta_2} = 0. \qquad (28)$$

5 Comparative Statics

In this section we conduct a comparative statics analysis of the value functions $\mathcal{V}$, V_2 in market 1 and 2, respectively as well as the exit threshold θ_{E_2}.

We first establish the convexity of V_2 which leads to the comparative statics with respect to σ.

Proposition 5 *The optimal return function V_2 is convex for both cases.*

The proof is stated in Appendix B. Next, we examine the comparative statics of $V_2(.)$ with respect to α_2, σ and $\theta_2(0)$.

Proposition 6 *The optimal return function V_2 is non-decreasing in α_2, in σ and $\theta_2(0)$ for both cases. If $\theta_2(0) > \theta_{E_2}$ then V_2 is strictly increasing in α_2.*

These results lead us to the comparative statics regarding the exit threshold with respect to α_2 and σ, stated in the following proposition.

Proposition 7 *The exit threshold θ_{E_2} is non-increasing in α_2, in σ and $\theta_2(0)$ for both cases. If $\theta_2(0) > \theta_{E_2}$ then θ_{E_2} is strictly decreasing in α_2.*

Based on the expressions for θ_{E_2}, in equations (5) and (6) for the first model and (22) for the benchmark model, we can further infer that:

Remark 1 *The exit threshold* θ_{E_2} *is inversely proportional to* γ *in both cases.*

Remark 2 *In the first model, for the case* $\theta_S < \theta_{E_2}$ *(i.e. exit out of production), for* K_2 *arbitrarily small or large,* θ_{E_2} *is arbitrarily large. For the other case, if* $\beta_3 > 2$, θ_{E_2} *is increasing with* K_2*, otherwise* θ_{E_2} *decreases with* K_2*.*

Now for an analysis of the value function of market 1, $\mathcal{V}(.)$:

Proposition 8 *The optimal return function* $\mathcal{V}$ *is convex for both cases.*

Proposition 9 *For all* $\theta \in \Re^+$*, the value function of the firm* $\mathcal{V}$ *is nondecreasing in* α_1*,* α_2 *and* σ*.*

—¿WRITE SOMETHING HERE¡—

Proposition 10 *The probability that the firm is investing rather than exiting, i.e. the probability that threshold* θ_I *is hit before* θ_{E_1}*, is given by*

$$P_I = \frac{\left(\frac{\theta(0)}{\theta_{E_1}}\right)^{1-\frac{2\alpha_1}{\sigma^2}} - 1}{\left(\frac{\theta_I}{\theta_{E_1}}\right)^{1-\frac{2\alpha_1}{\sigma^2}} - 1}. \tag{29}$$

Proposition 11 *Let* $T_{E_2} = \inf\{t : \theta_2(t) = \theta_{E_2} | \theta_2(0) = \theta_I\}$ *be the time that the firm exits the second market once it has invested. Then its expected value is given by*

$$E\left[T_{E_2}\right] = \begin{cases} \frac{\ln\left[\frac{\theta_I}{\theta_{E_2}}\right]}{\frac{1}{2}\sigma^2 - \alpha_2} & \text{if } \alpha_2 < \frac{1}{2}\sigma^2, \\ \infty & \text{otherwise.} \end{cases} \tag{30}$$

Proposition 12 *Let* T *denote the time until the company decides, optimally, to invest in a new technology or to exit the market (which one occurs first). Then*

$$E[T] = \frac{1}{\frac{1}{2}\sigma^2 - \alpha_1}\left(\ln\left[\frac{\theta(0)}{\theta_{E_1}}\right] - \frac{1 - \left(\frac{\theta(0)}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}}}{1 - \left(\frac{\theta_I}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}}} \ln\left[\frac{\theta_I}{\theta_{E_1}}\right]\right).$$

6 Results

6.1 Effect of increasing uncertainty on threshold

The standard real options result says that the investment threshold goes up with increased uncertainty reflecting the value of waiting. Tables 2 and 1 show for the considered investment decision, a different result that holds. The investment threshold first goes up with uncertainty but decreases for high values of uncertainty. The latter is caused by the existence of the exit option. The value of this option increases

Table 1: Benchmark Model. Parameter Values: $\alpha_1 = -0.02$, $\alpha_2 = -0.01$, $r = 0.1$, $c = 1$, $\mu = 0.2$, $\gamma = 1.2$, $F = 2$, $I = 100$. Thus $\theta_S = \frac{c}{\mu} = 5$.

σ	0.1	0.15	0.2	0.25	0.3	0.325
θ_I	6.74	7.09	7.42	7.62	7.29	6.27
θ_{E_1}	5.12	4.03	2.94	1.95	1.07	0.64
θ_{E_2}	1.94	1.68	1.41	1.11	0.75	0.5
Case	1	2	2	2	2	2
$P[invest]_{\theta_S}$	x	21.56%	35.24%	44.18%	55.21%	71.97%
$E[T_{E_2}]_{\theta(0)=\theta_I}$	83.03	67.76	55.35	46.70	41.35	40.2615

Table 2: Benchmark Model. Parameter Values: $\alpha_1 = -0.02$, $\alpha_2 = 0.01$, $r = 0.1$, $c = 1$, $\mu = 0.2$, $\gamma = 1.2$, $F = 2$, $I = 100$. Thus $\theta_S = \frac{c}{\mu} = 5$.

σ	0.1	0.125	0.15	0.175	0.2	0.25
θ_I	5.38	5.43	5.45	5.41	5.26	4.21
θ_{E_1}	4.09	3.60	3.10	2.58	2.08	1.07
θ_{E_2}	1.67	1.53	1.38	1.22	1.05	0.64
Case:	2	2	2	2	2	3
$P[invest]_{\theta_S}$	58.90%	66.88%	73.10%	79.71%	88.57%	x
$E[T_{E_2}]_{\theta(0)=\theta_I}$	∞	∞	1098.83	280.36	161.134	88.647

with uncertainty and when investing the firm gains this option to exit. As a result the value of investment increases with uncertainty and therefore the firm wants to invest sooner. A similar results was obtained by Kwon (2010). However, he used a Brownian motion with drift to model the uncertainty in the profit stream. As a result it is not clear what the relation is between the investment timing with the threshold. Modeling the uncertainty with a geometric Brownian motion allows us to give insight about the effect of uncertainty on the threshold as well as the timing given that the firm invests. See the expression for the expected time in Proposition 12. Table

However, when the investment decision not only involves timing but also capacity size has to be determined the investment threshold is again monotonically increasing in uncertainty. When uncertainty is high the firm invests in a larger capacity. This implies that the exit threshold for the second product is higher and thus the second product will be produced during a shorter time. This reduces the value of the investment so that the firm has less incentive to invest and it invest later.

6.2 Effect of increasing uncertainty on exit thresholds

From standard real options literature it is known that the exit threshold is decreasing with uncertainty (see e.g. Dixit (1989) and Kwon (2010)). The intuition is that the value of the option to resume production or investment in the innovative product is higher when uncertainty is larger and therefore, the firm wants to keep these options alive. However, Tables 3 and 4 show that in our framework the effect of uncertainty on the exit threshold is non-monotonic. The increasing part is caused by the effect of the capacity choice. Due to the inflexible production structure the firm is forced to produce up to capacity all the time. When uncertainty goes up the capacity size K_2^* and thus production is larger. This leads to considerable losses when θ is low. This induces the firm to exit before the θ is too low. The decreasing part is due to the reason already given, i.e. the value of keeping the options to resume production and/or investing in the innovative product alive increases with uncertainty.

6.3 Effect of old market production capacity

The decision of the firm to invest in the new market depends strongly on the capacity size in the old market. `Work in Progress`

6.4 When does suspension not occur?

When the value of the option to start again production is low, the firm will not use the suspension option but instead exit operations. This happens in cases that uncertainty is low and the drift is sufficiently low. In addition in case of considering suspension versus exit in the first market suspension does not occur when

Table 3: Parameter Values: $\alpha_1 = -0.02, \alpha_2 = -0.01, r = 0.1, c = 1, \mu = 1, \gamma = 1.2, K_1 = 2, \delta = 10, F = 2$, i.e. $\theta_{S_1} = \frac{c+K_1}{\mu} = 3$, $\theta(0) = \frac{\theta_I + \theta_{E_1}}{2}$.

σ	0.05	0.1	0.15	0.2	0.25	0.3
K_2	1.452	1.568	1.808	2.595	6.123	14.694
θ_I	3.546	3.668	3.954	5.071	9.676	18.76
θ_{E_1}	3.401	3.168	2.891	2.546	2.105	1.628
θ_{E_2}	2.195	1.989	1.832	1.860	1.941	1.630
Case	4	4	5	2	2	2
$E[T]$	0.173	0.533	1.078	2.883	8.370	13.304
P_I	41.73%	42.74%	43.14%	41.71%	39.34%	39.67%
$E[T_{E_2}]$	3.711	9.770	14.735	22.967	36.978	44.44

Table 4: Parameter Values: $\alpha_1 = -0.02, \alpha_2 = 0.01, r = 0.1, c = 1, \mu = 1, \gamma = 1.2, K_1 = 2, \delta = 10, F = 2$, i.e. $\theta_{S_1} = 3$, $\theta(0) = \frac{\theta_I + \theta_{E_1}}{2}$.

σ	0.05	0.1	0.15	0.2	0.25
K_2	1.439	1.515	1.661	1.966	3.549
θ_I	2.907	2.985	3.115	3.416	5.073
θ_{E_1}	2.829	2.686	2.460	2.183	1.855
θ_{E_2}	1.971	1.772	1.602	1.480	1.352
Case	6	6	5	2	2
$E[T]$	0.074	0.277	0.616	1.238	3.856
P_I	44.62%	44.75%	44.80%	44.49%	42.43%
$E[T_{E_2}]$	∞	∞	188.85	44.77	47.34

Table 5: Parameter Values: $\alpha_1 = -0.02, \alpha_2 = -0.01, r = 0.1, c = 1, \mu = 1, \gamma = 1.2, \sigma = 0.1, \delta = 10, F = 2$, $\theta(0) = \frac{\theta_I + \theta_{E_1}}{2}$.

K_1	0.5	1	1.5	2	2.5	3
K_2	1.571	1.631	1.632	1.568	1.514	1.513
θ_I	3.673	3.779	3.781	3.668	3.573	3.571
θ_{E_1}	3.125	3.041	3.071	3.168	3.229	3.229
θ_{E_2}	1.989	1.998	1.998	1.989	1.982	1.982
Case	4	4	4	4	5	6
$E[T]$	0.646	1.160	1.064	0.533	0.255	0.252
P_I	42.02%	39.36%	39.80%	42.74%	44.96%	44.99%
$E[T_{E_2}]$	10.77	14.48	13.87	9.77	6.75	6.71

Table 6: Parameter Values: $\alpha_1 = -0.02, \alpha_2 = 0.01, r = 0.1, c = 1, \mu = 1, \gamma = 1.2, \sigma = 0.1, \delta = 10, F = 2$, $\theta(0) = \frac{\theta_I + \theta_{E_1}}{2}$.

K_1	0.5	1	1.5	2	2.5	3
K_2	1.550	1.565	1.543	1.515	1.515	1.515
θ_I	3.037	3.059	3.027	2.985	2.985	2.985
θ_{E_1}	2.633	2.618	2.656	2.686	2.686	2.686
θ_{E_2}	1.775	1.777	1.774	1.772	1772	1.772
Case	4	4	4	6	6	6
$E[T]$	0.506	0.600	0.425	0.277	0.277	0.277
P_I	42.93%	42.30%	43.51%	44.75%	44.75%	44.75%
$E[T_{E_2}]$	∞	∞	∞	∞	∞	∞

Table 7: Parameter Values: $\alpha_1 = \alpha_2 = -0.02, r = 0.1, c = 0, \mu = \gamma = 1, \sigma = 0.1, \delta = 10, F = 2$, $\theta(0) = \frac{\theta_I + \theta_{E_1}}{2}$.

K_1	0.5	1	1.5	2	2.5	3
K_2	1.745	2.530	3.402	4.246	5.069	5.895
θ_I	5.131	6.957	8.977	10.924	12.812	14.701
θ_{E_1}	3.589	2.604	2.461	2.606	2.867	3.186
θ_{E_2}	2.511	2.885	3.466	4.078	4.647	5.195
Case	4	4	4	1	1	1
$E[T]$	3.052	18.524	28.369	32.98	35.13	36.22
P_I	33.10%	14.69%	10.36%	9.08%	8.53%	8.29%
$E[T_{E_2}]$	14.30	39.31	51.76	57.33	59.88	61.17

the option to invest in the new product is not attractive, i.e. γ is low and α_2 as well as σ are also low.

6.5 Constant capacity when investing out of suspension

Tables 8, 9 and 10 show that the capacity level is insensitive to γ when the firm invests out of the suspension region, i.e. Cases 3 and 6. This can be explained as follows. Investing out of the suspension region implies that the firm does not produce before investment. Therefore, the firm gives up the same current revenue when investing in the innovative product. Next, we want to argue that the firm invests such that also after the investment the revenues are the same. We do this by showing that the firm chooses the investment threshold such that the price as well as the price dynamics stay the same. From the table it can be obtained that the value of $\gamma\theta_I$ stays constant in γ. The latter characteristic implies that the output price at the moment of investment for the innovative product is the same in all situations. To consider the dynamics, we derive

$$dp = \alpha_2 \gamma \theta dt + \sigma \gamma \theta dz \tag{31}$$

from Equations 1 and 2. Hence the conclusion is that also the price dynamics stay the same for different γ. We conclude that in all situations the investment of δK_2 generates the same revenue stream in all situation. This explains why the chosen capacity level K_2 is also the same. The same reasoning explains why the capacity stays constant when changing K_1 while investing out of suspension, see Table 6 for $K_1 = 2, 2.5, 3$.

Table 8: Model from proposition 1: $\alpha_1 = -0.02, \alpha_2 = -0.01, r = 0.1, c = 1, \mu = 1, \sigma = 0.1, K_1 = 2, \delta = 10, F = 2$, i.e. $\theta_{S_1} = 3$, $\theta(0) = \frac{\theta_I + \theta_{E_1}}{2}$.

γ	1	1.5	2	3	5	10
K_2	2.152	1.513	1.513	1.513	1.513	1.513
θ_I	5.661	2.857	2.142	1.428	0.857	0.428
θ_{E_1}	3.460	2.583	1.938	1.292	0.775	0.388
θ_{E_2}	2.586	1.583	1.189	0.793	0.476	0.238
Case	4	6	6	6	6	6
$E[T]$	5.578	0.253	0.249	0.249	0.252	0.240
P_I	27.77%	44.98%	45.02%	45.02%	44.99%	45.12%
$E[T_{E_2}]$	32.82	6.72	6.67	6.67	6.70	6.54

Table 9: Model from proposition 1: $\alpha_1 = -0.02, \alpha_2 = 0.01, r = 0.1, c = 1, \mu = 1, \sigma = 0.1, K_1 = 2, \delta = 10, F = 2$, i.e. $\theta_{S_1} = 3$, $\theta(0) = \frac{\theta_I + \theta_{E_1}}{2}$.

γ	1	1.5	2	3	5	10
K_2	1.571	1.515	1.515	1.515	1.515	1.515
θ_I	3.682	2.388	1.791	1.194	0.716	0.358
θ_{E_1}	3.161	2.149	1.612	1.075	0.645	0.322
θ_{E_2}	2.133	1.418	1.063	0.709	0.425	0.213
Case	4	6	6	6	6	6
$E[T]$	0.577	0.277	0.276	0.274	0.272	0.280
P_I	42.45%	44.75%	44.76%	44.78%	44.80%	44.73%
$E[T_{E_2}]$	∞	∞	∞	∞	∞	∞

Table 10: Model from proposition 1: $\alpha_1 = -0.02, \alpha_2 = 0.01, r = 0.1, c = 1, \mu = 1, \sigma = 0.2, K_1 = 2, \delta = 10, F = 2$, i.e. $\theta_{S_1} = 3$, $\theta(0) = \frac{\theta_I + \theta_{E_1}}{2}$.

γ	1	1.5	2	3	5	10
K_2	2.682	1.920	1.920	1.920	1.920	1.920
θ_I	5.240	2.683	2.013	1.342	0.805	0.403
θ_{E_1}	2.539	1.748	1.311	0.874	0.525	0.262
θ_{E_2}	1.890	1.179	0.884	0.589	0.354	0.177
Case	2	3	3	3	3	3
$E[T]$	3.178	1.134	1.136	1.136	1.129	1.146
P_I	41.32%	44.72%	44.72%	44.72%	44.74%	44.70%
$E[T_{E_2}]$	72.46	42.85	42.88	42.88	42.74	43.06

Table 11: Parameter Values: $\alpha_1 = \alpha_2 = -0.02, r = 0.1, c = 1, \mu = 1, \gamma = 1.2, \sigma = 0.1, \delta = 10, F = 2$, $\theta(0) = \frac{\theta_I + \theta_{E_1}}{2}$.

K_1	0.5	1	1.5	2	2.5	3
K_2	1.583	1.684	1.729	1.642	1.539	1.512
θ_I	3.968	4.160	4.244	4.079	3.886	3.835
θ_{E_1}	3.343	3.198	3.205	3.333	3.459	3.478
θ_{E_2}	2.061	2.079	2.089	2.071	2.055	2.052
Case	4	4	4	4	5	6
$E[T]$	0.726	1.686	1.915	1.005	0.337	0.238
P_I	41.54%	37.25%	36.44%	40.09%	44.22%	45.14%
$E[T_{E_2}]$	6.86	10.52	11.23	8.08	4.66	3.91

7 Conclusions

WORK IN PROGRESS

A Additional Model Details

A.1 Market 2

In order to derive the constant parameters (D_1, D_2, D_3 and G_1) of the value function and the exit thresholds for the two cases, we study the decision to exit the market. Denoting the exit threshold by θ_{E2}, we can write down the following value matching and smooth pasting conditions for the case $\theta_{E_2} < \theta_S$:

$$\begin{aligned} V_{2,2}(\theta)|_{\theta=\theta_{E_2}} &= 0, & (32)\\ \left.\frac{\partial V_{2,2}(\theta_2)}{\partial\theta}\right|_{\theta=\theta_{E_2}} &= 0, & (33)\\ V_{2,1}(\theta)|_{\theta=\frac{K_2}{\gamma}} &= V_{2,2}(\theta)|_{\theta=\frac{K_2}{\gamma}}, & (34)\\ \left.\frac{\partial V_{2,1}(\theta)}{\partial\theta}\right|_{\theta=\frac{K_2}{\gamma}} &= \left.\frac{\partial V_{2,2}(\theta)}{\partial\theta}\right|_{\theta=\frac{K_2}{\gamma}}, & (35)\end{aligned}$$

where

$$V_2(\theta, K_2) = \begin{cases} \frac{\gamma K_2}{r-\alpha_2}\theta - \frac{K_2^2+F}{r} + D_1\theta^{\beta_4} & \text{for } \theta > \frac{K_2}{\gamma} & =: V_{2,1}(\theta), \\ -\frac{F}{r} + D_2\theta^{\beta_3} + D_3\theta^{\beta_4} & \text{for } \theta \le \frac{K_2}{\gamma} & =: V_{2,2}(\theta)). \end{cases} \quad (36)$$

Equations (77) and (78) account for the fact that the value function has to be smooth in $\theta = \frac{K_2}{\gamma}$. Solving equations (75) to (78) one can easily derive the exit threshold θ_{E_2} and the expressions for the parameters D_1, D_2 and D_3:

$$\begin{aligned} \theta_{E_2} &= \left(\frac{\beta_4\alpha_2 - r}{\beta_4(r-\alpha_2)}\right)^{\frac{-1}{\beta_3}} F^{\frac{1}{\beta_3}} \frac{1}{\gamma} K_2^{1-\frac{2}{\beta_3}}, \\ D_1 &= \frac{r-\beta_3\alpha_2}{r(r-\alpha_2)(\beta_3-\beta_4)}\gamma^{\beta_4} K_2^{2-\beta_4} + D_3, \\ D_2 &= \frac{r-\beta_4\alpha_2}{r(r-\alpha_2)(\beta_3-\beta_4)}\gamma^{\beta_3} K_2^{2-\beta_3}, \\ D_3 &= \theta_{E_2}^{-\beta_4} \frac{\beta_3}{\beta_3-\beta_4}\frac{F}{r}. \end{aligned}$$

For the second case, i.e. $\theta_{E_2} > \theta_S$, the value matching and smooth pasting conditions are given by

$$\begin{aligned} V_2(\theta)|_{\theta=\theta_{E_2}} &= 0, & (37)\\ \left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_2}} &= 0. & (38)\end{aligned}$$

Solving this equation system one obtains

$$\theta_{E_2} = \frac{r-\alpha_2}{r\gamma}\frac{\beta_4}{\beta_4-1}\left(K_2+\frac{F}{K_2}\right),$$
$$G_1 = -\frac{\gamma K_2}{r-\alpha_2}\frac{1}{\beta_4}\left(\frac{r-\alpha_2}{r\gamma}\frac{\beta_4}{\beta_4-1}\left(K_2+\frac{F}{K_2}\right)\right)^{1-\beta_4}.$$

A.2 Market 1

In order to derive the constant parameters and the investment as well as the exit threshold (θ_I and θ_{E_1}) we use the value matching and smooth pasting conditions. Those are given for the three different cases in the following.

A.2.1 Case 1 - $\theta_{E_1} > \theta_{S_1}$ and $\theta_{E_2} \leq \theta_{S_2}$

The value matching and smooth pasting conditions for case 1 are given by:

$$V_1(\theta)|_{\theta=\theta_{E_1}} = 0 \tag{39}$$
$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_1}} = 0 \tag{40}$$
$$V_1(\theta)|_{\theta=\theta_I} = V_2(\theta)|_{\theta=\theta_I} - \delta K_2 \tag{41}$$
$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_I} = \left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_I} \tag{42}$$

Equations (82) and (83) lead to the following expressions for the parameters A_1 and A_2:

$$A_1 = \frac{a_1(\beta_2-1)\theta_{E_1}^{1-\beta_1} - a_2\beta_2\theta_{E_1}^{-\beta_1}}{\beta_1-\beta_2}$$
$$A_2 = \frac{a_1(\beta_1-1)\theta_{E_1}^{1-\beta_2} - a_2\beta_1\theta_{E_1}^{-\beta_2}}{\beta_2-\beta_1}$$

From equations (84) and (85) (alongside the first two) we obtain equations (8) and (9) which implicity give the investment and exit threshold θ_I and θ_{E_1}.

A.2.2 Case 2 - $\theta_{E_1} < \theta_{S_1} < \theta_I$ and $\theta_{E_2} \leq \theta_{S_2}$

For case 2 the corresponding value matching and smooth pasting conditions are given by:

$$\begin{aligned}
V_{1,2}(\theta)|_{\theta=\theta_{E_1}} &= 0 && (43)\\
\left.\frac{\partial V_{1,2}(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_1}} &= 0 && (44)\\
V_{1,1}(\theta)|_{\theta=\frac{c+K_1}{\mu}} &= V_{1,2}(\theta)|_{\theta=\frac{c+K_1}{\mu}} && (45)\\
\left.\frac{\partial V_{1,1}(\theta)}{\partial\theta}\right|_{\theta=\frac{c+K_1}{\mu}} &= \left.\frac{\partial V_{1,2}(\theta)}{\partial\theta}\right|_{\theta=\frac{c+K_1}{\mu}} && (46)\\
V_{1,1}(\theta)|_{\theta=\theta_I} &= V_2(\theta)|_{\theta=\theta_I} - \delta K_2 && (47)\\
\left.\frac{\partial V_{1,1}(\theta)}{\partial\theta}\right|_{\theta=\theta_I} &= \left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_I} && (48)
\end{aligned}$$

Equations (86) through (89) leads to the following expressions for the parameters $B_1 - B_4$:

$$\begin{aligned}
B_1 &= \frac{-a_1(\beta_2-1)\left(\frac{c+K_1}{\mu}\right)^{1-\beta_1} + \beta_2\frac{cK_1+K_1^2}{r}\left(\frac{c+K_1}{\mu}\right)^{-\beta_1}}{\beta_2-\beta_1} + B_3\\
B_2 &= \frac{-a_1(\beta_1-1)\left(\frac{c+K_1}{\mu}\right)^{1-\beta_2} + \beta_1\frac{cK_1+K_1^2}{r}\left(\frac{c+K_1}{\mu}\right)^{-\beta_2}}{\beta_1-\beta_2} + B_4\\
B_3 &= \theta_{E_1}^{-\beta_1}\frac{\beta_2}{\beta_2-\beta_1}\frac{F}{r}\\
B_4 &= \theta_{E_1}^{-\beta_2}\frac{\beta_1}{\beta_1-\beta_2}\frac{F}{r}
\end{aligned}$$

The equations (10) and (11) are obtained from (90) and (91), where the first explicitly gives an expression for θ_{E_1} and the latter an implicit expression for θ_I.

A.2.3 Case 3 - $\theta_{E_1} < \theta_I < \theta_{S_1}$ and $\theta_{E_2} \leq \theta_{S_2}$

The value matching and smooth pasting conditions for this case are given by

$$\begin{aligned}
V_1(\theta)|_{\theta=\theta_{E_1}} &= 0 && (49)\\
\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_1}} &= 0 && (50)\\
V_1(\theta)|_{\theta=\theta_I} &= V_2(\theta)|_{\theta=\theta_I} - \delta K_2 && (51)\\
\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_I} &= \left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_I} && (52)
\end{aligned}$$

As before, from the first two equations (92) and (93) result in the following expressions for the parameters C_1 and C_2, respectively.

$$C_1 = \theta_{E_1}^{-\beta_1} \frac{\beta_2}{\beta_2 - \beta_1} \frac{F}{r}$$
$$C_2 = \theta_{E_1}^{-\beta_2} \frac{\beta_1}{\beta_1 - \beta_2} \frac{F}{r}$$

The equations (12) and (13) are obtained from (94) and (95), where the first explicitly gives an expression for θ_{E_1} and the latter an implicit expression for θ_I.

For the other three cases we have the same value matchings and smooth pastings, the only change is on V_2, therefore the values of A, B and C do not change. Whereas the expressions for θ_I and θ_{E_1} change accordingly, but are obtained like in the previous cases.

A.3 Market 2

In order to derive the constant parameter A we study the decision to exit the market, we can write down the following value matching and smooth pasting conditions:

$$V_2(\theta) = \frac{\gamma^2\theta^2}{4(r - 2\alpha_2 - \sigma^2)} - \frac{F}{r} + A\theta^{\beta_4}.$$

$$V_2(\theta)|_{\theta=\theta_{E_2}} = 0, \tag{53}$$
$$\left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_2}} = 0. \tag{54}$$

Solving equations (96) to (97) one can easily derive the exit threshold θ_{E_2} and the expressions for the parameter A:

$$\theta_{E_2} = \sqrt{\frac{F}{\gamma^2 r} 4(r - 2\alpha_2 - \sigma^2)\left(\frac{\beta_4}{\beta_4 - 2}\right)}$$
$$A = \theta_{E_2}^{-\beta_4} \frac{F}{r}\left(\frac{2}{2 - \beta_4}\right).$$

A.4 Market 1

A.4.1 Case 1 - $\theta_{E_1} > \frac{c}{\mu}$

Value matching and smooth pasting for this case give:

$$V_1(\theta)|_{\theta=\theta_{E_1}} = 0 \tag{55}$$
$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_1}} = 0 \tag{56}$$
$$V_1(\theta)|_{\theta=\theta_I} = V_2(\theta)|_{\theta=\theta_I} - I \tag{57}$$
$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_I} = \left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_I} \tag{58}$$

Equations (98) and (99) lead to the following expressions for the parameters A_1 and A_2:

$$A_1 = \frac{b_1(\beta_2-2)\theta_{E_1}^{2-\beta_1} - b_2(\beta_2-1)\theta_{E_1}^{1-\beta_1} + b_3\beta_2\theta_{E_1}^{-\beta_1}}{\beta_1-\beta_2}$$
$$A_2 = \frac{b_1(\beta_1-2)\theta_{E_1}^{2-\beta_2} - b_2(\beta_1-1)\theta_{E_1}^{1-\beta_2} + b_3\beta_1\theta_{E_1}^{-\beta_2}}{\beta_2-\beta_1}$$

From equations (100) and (101) (alongside the first two) we obtain equations (23) and (24) which implicity give the investment and exit threshold θ_I and θ_{E_1}.

A.4.2 Case 2 - $\theta_{E_1} < \frac{c}{\mu} < \theta_I$

Value matching and smooth pasting for this case give:

$$V_{1,2}(\theta)|_{\theta=\theta_{E_1}} = 0 \quad (59)$$
$$\left.\frac{\partial V_{1,2}(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_1}} = 0 \quad (60)$$
$$V_{1,1}(\theta)|_{\theta=\frac{c}{\mu}} = V_{1,2}(\theta)|_{\theta=\frac{c}{\mu}} \quad (61)$$
$$\left.\frac{\partial V_{1,1}(\theta)}{\partial\theta}\right|_{\theta=\frac{c}{\mu}} = \left.\frac{\partial V_{1,2}(\theta)}{\partial\theta}\right|_{\theta=\frac{c}{\mu}} \quad (62)$$
$$V_{1,1}(\theta)|_{\theta=\theta_I} = V_2(\theta)|_{\theta=\theta_I} - I \quad (63)$$
$$\left.\frac{\partial V_{1,1}(\theta)}{\partial\theta}\right|_{\theta=\theta_I} = \left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_I} \quad (64)$$

Equations (102) through (105) leads to the following expressions for the parameters $B_1 - B_4$:

$$B_1 = \frac{b_1(\beta_2-2)\left(\frac{c}{\mu}\right)^{2-\beta_1} - b_2(\beta_2-1)\left(\frac{c}{\mu}\right)^{1-\beta_1} + \beta_2(b_3+b_5)\left(\frac{c}{\mu}\right)^{-\beta_1}}{\beta_1-\beta_2} + B_3$$
$$B_2 = \frac{b_1(\beta_1-2)\left(\frac{c}{\mu}\right)^{2-\beta_2} - b_2(\beta_1-1)\left(\frac{c}{\mu}\right)^{1-\beta_2} + \beta_1(b_3+b_5)\left(\frac{c}{\mu}\right)^{-\beta_2}}{\beta_2-\beta_1} + B_4$$
$$B_3 = \theta_{E_1}^{-\beta_1}\frac{\beta_2}{\beta_2-\beta_1}b_5$$
$$B_4 = \theta_{E_1}^{-\beta_2}\frac{\beta_1}{\beta_1-\beta_2}b_5$$

The equations (25) and (26) are obtained from (106) and (107), where the first explicitly gives an expression for θ_{E_1} and the latter an implicit expression for θ_I.

A.4.3 Case 3 - $\theta_{E_1} < \theta_I < \frac{c}{\mu}$

The value matching and smooth pasting conditions for this case are given by

$$V_1(\theta)|_{\theta=\theta_{E_1}} = 0 \quad (65)$$

$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_1}} = 0 \quad (66)$$

$$V_1(\theta)|_{\theta=\theta_I} = V_2(\theta)|_{\theta=\theta_I} - I \quad (67)$$

$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_I} = \left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_I} \quad (68)$$

As before, from the first two equations (92) and (93) result in the following expressions for the parameters C_1 and C_2, respectively.

$$C_1 = \theta_{E_1}^{-\beta_1}\frac{\beta_2}{\beta_2-\beta_1}b_5$$
$$C_2 = \theta_{E_1}^{-\beta_2}\frac{\beta_1}{\beta_1-\beta_2}b_5$$

The equations (27) and (28) are obtained from (110) and (111), where the first explicitly gives an expression for θ_{E_1} and the latter an implicit expression for θ_I.

B Proofs

B.1 Proof of Proposition 1

Now assume that $V_1(.)$ in that proposition is a candidate for the optimal value function, with three cases. In the following we verify that $V_1(.)$ it indeed satisfies all the sufficient conditions for being the optimal value function specified in Theorem 10.4.1 of **?**. Oksendal's $\phi(.)$, $f(.)$ and $g(.)$ are here given by $V_1(.)$, $\Pi_1(.)$ and $V_2(\theta, K_2) - I(K_2)$, respectively. And we have that $V = \Re_0^+$, $D = (\theta_{E_1}, \theta_I)$, therefore $\partial D = \{\theta_{E_1}, \theta_I\}$.

Conditions (iii), (iv), (viii) and (ix) hold trivially because θ follows a geometric Brownian motion. $V_1(.)$ is continuous differentiable in V since we impose the value matching and smooth pasting conditions in ∂D, alongside θ_{S_1} (see equations (82) through (95)), which relates with (i). Furthermore, $V_1(.)$ is twice continuously differentiable except at $\partial D \cup \{\theta_{S_1}\}$. Since $V_1(.)$ is a polynomial in θ, the second order derivatives of $V_1(.)$ are finite near ∂D, which checks condition (v).

Moreover, we introduce the partial differential operator L applied to the process $\{\theta(t); t \geq 0\}$:$L = \frac{\partial}{\partial t} + \alpha(\theta)\theta\frac{\partial}{\partial\theta} + \frac{1}{2}\sigma^2\theta^2\frac{\partial^2}{\partial\theta^2}$. But since the time-dependence of the return function is only throufh the discount factor e^{-rt}, the infinitesimal generator can be replaced by:

$$L = -r + \alpha_1\theta\frac{\partial}{\partial\theta} + \frac{1}{2}\sigma^2\theta^2\frac{\partial^2}{\partial\theta^2}$$

To obtain (ii) we use similar reasoning as **?**. To achieve (vi) and (vii) we consider one case of V_1, while for the other cases similar calculations will apply.

$$V_1(\theta) = \begin{cases} \frac{\mu K_1 \theta}{r-\alpha_1} - \frac{cK_1+K_1^2+F}{r} + B_1\theta^{\beta_1} + B_2\theta^{\beta_2} & \text{for } \theta \geq \frac{c+K_1}{\mu} \\ -\frac{F}{r} + B_3\theta^{\beta_1} + B_4\theta^{\beta_2} & \text{for } \theta < \frac{c+K_1}{\mu} \end{cases}$$

$$LV_1(\theta) = \begin{cases} -\mu K_1\theta + (cK_1 + K_1^2 + F) & \text{for } \theta \geq \frac{c+K_1}{\mu} \\ F & \text{for } \theta < \frac{c+K_1}{\mu} \end{cases}$$

Therefore condition (vii) holds since $LV_1(\theta) + \Pi_1(\theta) = 0$. For condition (vi) we proceed similarly as ?. □

B.2 Proof of Proposition 2

All the calculus needed are stated in Appendix A.2. □

B.3 Proof of Proposition 3

The proof is similar to the proof of Proposition 1, therefore we omit it. □

B.4 Proof of Proposition 2

All the calculus needed are stated in Appendix A.4. □

B.5 Proof of Proposition 5

First we show that Π_2 is convex for both cases.

For the case "Model - Capacity, Timing" the profit function is given by:

$$\Pi_2(\theta) = \begin{cases} \gamma K_2\theta - K_2^2 - F & \text{for } \theta > \frac{K_2}{\gamma}, \\ -F & \text{for } \theta \leq \frac{K_2}{\gamma}. \end{cases}$$

Let $\theta_x = x\theta_1 + (1-x)\theta_2$, where $x \in (0,1)$. If $\theta_1, \theta_2 > (<)\frac{K_2}{\gamma}$ then the convexity obviously holds. For the case $\theta_1 \leq \frac{K_2}{\gamma} \leq \theta_2$ then:

- If $\theta_x \leq \frac{K_2}{\gamma}$:

$$\Pi_2(\theta_x) \leq x\Pi_2(\theta_1) + (1-x)\Pi_2(\theta_2)$$
$$\Leftrightarrow -F \leq x(-F) + (1-x)(\gamma K_2\theta_2 - K^2 - F) \Leftrightarrow \frac{K_2}{\gamma} \leq \theta_2$$

- If $\theta_x > \frac{K_2}{\gamma}$:

$$\Pi_2(\theta_x) \leq x\Pi_2(\theta_1) + (1-x)\Pi_2(\theta_2)$$
$$\Leftrightarrow \gamma K_2(x\theta_1 + (1-x)\theta_2) - K^2 - F \leq x(-F) + (1-x)(\gamma K_2\theta_2 - K^2 - F) \Leftrightarrow \theta_1 \leq \frac{K_2}{\gamma}$$

For the Benchmark model, the profit function is given by

$$\Pi_2(\theta) = \frac{\gamma^2\theta^2}{4} - F.$$

which is obviously convex.

Taking into account that in both cases the profit function $\Pi(.)$ is convex for every θ_1, θ_2 and $x \in (0,1)$, we obtain the following inequality:

$$\begin{aligned}
V_2(\theta_x) &= \sup_{\tilde{\tau}} \mathbb{E}^{\theta_x}\left[\int_0^{\tilde{\tau}} e^{-rt}\Pi_2(\theta_2(t))dt\right], \\
&= \sup_{\tilde{\tau}} \mathbb{E}^{\theta_x}\left[\int_0^{\tilde{\tau}} e^{-rt}\Pi_2\left(\theta_x e^{\left(\left(\alpha_2-\frac{\sigma^2}{2}\right)t+\sigma z_t\right)}\right)dt\right], \\
&\leq x\sup_{\tilde{\tau}} \mathbb{E}^{\theta_1}\left[\int_0^{\tilde{\tau}} e^{-rt}\Pi_2\left(\theta_1 e^{\left(\left(\alpha_2-\frac{\sigma^2}{2}\right)t+\sigma z_t\right)}\right)dt\right] + (1-x)\sup_{\tilde{\tau}} \mathbb{E}^{\theta_2}\left[\int_0^{\tilde{\tau}} e^{-rt}\Pi_2\left(\theta_2 e^{\left(\left(\alpha_2-\frac{\sigma^2}{2}\right)t+\sigma z_t\right)}\right)dt\right], \\
&= xV_2(\theta_1) + (1-x)V_2(\theta_2), \qquad (69)
\end{aligned}$$

which concludes our proof. □

B.6 Proof of Proposition 6

Let $\mu > 0$ and denote by $V_2(\theta, \alpha_2)$ the value function V_2 with the dependence on the drift of the process here denoted by α_2.

$$\begin{aligned}
V_2(\theta, \alpha_2) &= \sup_{\tilde{\tau}} \mathbb{E}^{\theta_2}\left[\int_0^{\tilde{\tau}} e^{-rt}\Pi_2\left(\theta e^{\left(\left(\alpha_2-\frac{\sigma^2}{2}\right)t+\sigma z_t\right)}\right)dt\right], \\
&\leq \sup_{\tilde{\tau}} \mathbb{E}^{\theta_2}\left[\int_0^{\tilde{\tau}} e^{-rt}\Pi_2\left(\theta e^{\left(\left(\alpha_2+\mu-\frac{\sigma^2}{2}\right)t+\sigma z_t\right)}\right)dt\right], \qquad (70) \\
&\leq V_2(\theta, \alpha_2+\mu). \qquad (71)
\end{aligned}$$

Where inequality (70) follows from the fact that $e^{\left(\left(\alpha_2-\frac{\sigma^2}{2}\right)+\sigma z_t\right)} < e^{\left(\left(\alpha_2+\mu-\frac{\sigma^2}{2}\right)+\sigma z_t\right)}$ with probability 1 and Π_2 is non-decreasing in θ. The inequality becomes strict if $\tilde{\tau} > 0$ (i.e. $\theta_2(0) > \theta_{E_2}$). Moreover, as $\tilde{\tau}$ is the optimal stopping time for the problem with drift α_2 then it is suboptimal for the problem with drift $\alpha_2 + \mu$, which proves inequality (71).

Concerning the non-decreasing behavior of V_2 as a function of the volatility σ, we refer to **?**, page 273, where in a note he refers that for convex contract functions the option price, when the stock price follows a geometric Brownian motion, is non-decreasing in the volatility. This result holds in our case as V_2 is convex (see Proposition 5).

Finally, V_2 is non-decreasing in $\theta_2(0)$ because Π_2 is also non-decreasing in θ. □

B.7 Proof of Proposition 7

Noting that $\theta_{E_2} = \inf\{\theta : V_2(\theta) > 0\}$ the result follows in view of the last proposition. Since V_2 increases with the increase of α_2, σ or $\theta_2(0)$, its graphic will rise, and therefore zero will slide to the left, decreasing with the increase of any of those variables. □

B.8 Proof of Proposition 8

In the following we show that the profit function in region 1, $\Pi_1(.)$, is a convex function.

For the case "Model - Capacity, Timing" the profit function is given by:

$$\Pi_1(\theta) = \begin{cases} \mu K_1\theta - (cK_1 + K_1^2 + F) & \text{for } \theta > \frac{c+K_1}{\mu}, \\ -F & \text{for } \theta \leq \frac{c+K_1}{\mu}. \end{cases}$$

Let $\theta_x = x\theta_1 + (1-x)\theta_2$, where $x \in (0,1)$. If $\theta_1, \theta_2 > (<)\frac{c+K_1}{\mu}$ then the convexity obviously holds. For the case $\theta_1 \leq \frac{c+K_1}{\mu} \leq \theta_2$ it holds that

- if $\theta_x \leq \frac{c}{\mu}$:

$$\begin{aligned} &\Pi_1(\theta_x) \leq x\Pi_1(\theta_1) + (1-x)\Pi_1(\theta_2) \\ \Leftrightarrow& -F \leq x(-F) + (1-x)(\mu K_1\theta_2 - (cK_1 + K_1^2 + F)) \\ \Leftrightarrow& \frac{c+K_1}{\mu} \leq \theta_2 \end{aligned}$$

- If $\theta_x > \frac{c}{\mu}$:

$$\begin{aligned} &\Pi_1(\theta_x) \leq x\Pi_1(\theta_1) + (1-x)\Pi_1(\theta_2) \\ \Leftrightarrow& \mu K_1(x\theta_1 + (1-x)\theta_2) - (cK_1 + K_1^2 + F) \leq x(-F) + (1-x)(\gamma K_2\theta_2 - K_1^2 - F) \\ \Leftrightarrow& \theta_1 \leq \frac{c+K_1}{\mu} \end{aligned}$$

which shows the convexity for the remaining cases.

For the Benchmark model, the profit function is given by:

$$\Pi_1(\theta) = \begin{cases} \left(\frac{\mu\theta - c}{2}\right)^2 - F & \text{for } \theta > \frac{c}{\mu}, \\ -F & \text{for } \theta \leq \frac{c}{\mu}. \end{cases}$$

Let $\theta_x = x\theta_1 + (1-x)\theta_2$, where $x \in (0,1)$. If $\theta_1, \theta_2 > (<)\frac{c}{\mu}$ then the convexity obviously holds. For the case $\theta_1 \leq \frac{c}{\mu} \leq \theta_2$ it holds that

- if $\theta_x \leq \frac{c}{\mu}$:

$$\begin{aligned}&\Pi_1(\theta_x) \leq x\Pi_1(\theta_1) + (1-x)\Pi_1(\theta_2)\\ \Leftrightarrow &- F \leq x(-F) + (1-x)\left(\left(\frac{\mu\theta - c}{2}\right)^2 - F\right)\\ \Leftrightarrow &\frac{c}{\mu} \leq \theta_2\end{aligned}$$

- if $\theta_x > \frac{c}{\mu}$:

$$\begin{aligned}&\Pi_1(\theta_x) \leq x\Pi_1(\theta_1) + (1-x)\Pi_1(\theta_2)\\ \Leftrightarrow &\left(\frac{\mu(x\theta_1 + (1-x)\theta_2) - c}{2}\right)^2 - F \leq x(-F) + (1-x)\left(\left(\frac{\mu\theta_2 - c}{2}\right)^2 - F\right)\\ \Leftrightarrow &\theta_1 \leq \frac{c}{\mu}\end{aligned}$$

which shows the convexity for the remaining cases.

In a next step, we define

$$F(\theta_1) = \left[\int_0^\tau e^{-rt}\Pi_1\left(\theta_1(t)\right)dt \mid \theta_1(0) = \theta_1\right].$$

Then $F(\theta_1)$ is a convex function in θ_1 by the same reasoning that we used for inequality (69).

Finally, taking into account that the maximization of V_2 over K_2 preserves the convexity of the function, as well as the maximum and the sum of two convex functions, then

$$\mathcal{V}(\theta_1) = \sup_{\tilde{\tau}} \mathbb{E}^{\theta_1}\left[\int_0^{\tilde{\tau}} e^{-rt}\Pi_1(\theta_1(t))dt + e^{-r\tilde{\tau}} \max\left\{0, \max_{K_2}\left(V_2(\theta_1(\tilde{\tau}), K_2) - I(K_2)\right)\right\}\right],$$

where $I(K_2)$ is δK_2 in the first model and I in the Benchmark model, is also a convex function. □

B.9 Proof of Proposition 9

By proposition 6, V_2 is nondecreasing in α_2, and since α_2 affects only market 2, $\mathcal{V}$ is also nondecreasing in α_2.

Denote by $\mathcal{V}(\theta, \alpha_1)$ the value function $\mathcal{V}$ with the dependence on the drift of the process in market 1, here denoted by α_1. Let $\mu > 0$ such that $\alpha_1 + \mu < \alpha_2$.

$$\begin{aligned}
\mathcal{V}(\theta(0),\alpha_1) = &\sup_{\tilde{\tau}_{(\alpha_1)}} \mathbb{E}^{\theta}\left[\int_0^{\tilde{\tau}_{(\alpha_1)}} e^{-rt}\Pi_1\left(\theta(0)e^{\left(\left(\alpha_1-\frac{\sigma^2}{2}\right)t+\sigma z_t\right)}\right)dt\right.\\
&\left.+\, e^{-r\tilde{\tau}_{(\alpha_1)}}\max\left\{0,\max_{K_2}\left(V_2(\theta(\tilde{\tau}_{(\alpha_1)}),\alpha_2)-I(K_2)\right)\right\}\right]\\
&\leq \sup_{\tilde{\tau}_{(\alpha_1)}} \mathbb{E}^{\theta}\left[\int_0^{\tilde{\tau}_{(\alpha_1)}} e^{-rt}\Pi_1\left(\theta(0)e^{\left(\left(\alpha_1+\mu-\frac{\sigma^2}{2}\right)t+\sigma z_t\right)}\right)dt\right. && (72)\\
&\left.+\, e^{-r\tilde{\tau}_{(\alpha_1)}}\max\left\{0,\max_{K_2}\left(V_2(\theta(\tilde{\tau}_{(\alpha_1)})e^{\mu\tilde{\tau}_{(\alpha_1)}},\alpha_2)-I(K_2)\right)\right\}\right] && (73)\\
&\leq \mathcal{V}(\theta(0),\alpha_1+\mu) && (74)
\end{aligned}$$

where in (72) we use the fact that Π_1 is non-decreasing. We showed before that V_2 is increasing with the initial value $\theta_2(0)$, which proves (73) of the inequality. Finally in (74) the sub-optimality of $\tilde{\tau}_{(\alpha_1)}$ is used.

In order to prove the behavior of $\mathcal{V}$ as a function of σ, we follow Theorem 4 of **?**. Consider $d(x) = (r-\alpha_2)x$, which is an increasing function on $\Re^+$; then all the conditions of Theorem 4 of **?** are satisfied.[1]. Let $\nu(\theta_0) = E^{\theta_0}\left[e^{-r\tilde{\tau}}f(\theta(\tilde{\tau}))\right]$, where in our case f denotes the return of stopping at $\tilde{\tau}$ (which includes the return of $\max\{0,V_2\}$). Since $\max\{0,V_2\}$ is non-decreasing in σ, as proved before, we conclude based on Theorem 4 that $\mathcal{V}$ is a non-decreasing function of the volatility parameter σ. □

B.10 Proof of Proposition 10

Let $\theta(t)$ denote the demand level at time t, with $t < \tau_1$, so that the drift coefficient of the diffusion equation of θ is α_1. Therefore it follows that $\theta(t)$ is given by:

$$\theta(t) = \theta(0)\exp\left\{\left(\alpha_1 - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right\}$$

$$\begin{aligned}
\theta(t) > \theta_I \Leftrightarrow W(t) > \frac{1}{\sigma}\ln\left(\frac{\theta_I}{\theta(0)}\right) - \frac{1}{\sigma}\left(\alpha_1 - \frac{\sigma^2}{2}\right)t\\
\theta(t) < \theta_{E_1} \Leftrightarrow W(t) < \frac{1}{\sigma}\ln\left(\frac{\theta_{E_1}}{\theta(0)}\right) - \frac{1}{\sigma}\left(\alpha_1 - \frac{\sigma^2}{2}\right)t
\end{aligned}$$

We wish to have $\theta(t) > \theta_I$ before $\theta(t) < \theta_{E_1}$:

By theorem 4.1 of **?**, we have that if Y(t) is a Wiener Process, if $\gamma_1 > 0, \gamma_2 < 0, \delta_1 = \delta_2 \neq 0$, then the probability that $Y(t) \geq \gamma_1 + \delta_1 t$ for a smaller t than any t for which $Y(t) \leq \gamma_2 + \delta_2 t$ is:

$$P_I = \frac{e^{-2\gamma_2\delta_1} - 1}{e^{2(\gamma_1-\gamma_2)\delta_1} - 1}$$

For our case we have that

[1] In **?**, the function d is denoted by θ

$$\gamma_1 = \frac{1}{\sigma}\ln\left(\frac{\theta_I}{\theta(0)}\right) > 0$$
$$\gamma_2 = \frac{1}{\sigma}\ln\left(\frac{\theta_{E_1}}{\theta(0)}\right) < 0$$
$$\delta_1 = -\frac{1}{\sigma}\left(\alpha_1 - \frac{\sigma^2}{2}\right)$$

Therefore:

$$e^{-2\gamma_2\delta_1} = \exp\left\{\ln\left(\frac{\theta_{E_1}}{\theta(0)}\right)\left(2\frac{\alpha_1}{\sigma^2} - 1\right)\right\} = \left(\frac{\theta_{E_1}}{\theta(0)}\right)^{2\frac{\alpha_1}{\sigma^2}-1}$$
$$e^{2(\gamma_1-\gamma_2)\delta_1} = \exp\left\{-\ln\left(\frac{\theta_I}{\theta(0)}/\frac{\theta_{E_1}}{\theta(0)}\right)\left(2\frac{\alpha_1}{\sigma^2} - 1\right)\right\} = \left(\frac{\theta_{E_1}}{\theta_I}\right)^{2\frac{\alpha_1}{\sigma^2}-1}$$

So

$$P_I = \frac{\left(\frac{\theta(0)}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}} - 1}{\left(\frac{\theta_I}{\theta_{E_1}}\right)^{1-2\frac{\alpha_1}{\sigma^2}} - 1}$$

□

B.11 Proof of Proposition 11

Note that

$$T_{E_2} = \inf\left\{t : z_t + \frac{1}{\sigma}\left(\frac{\sigma^2}{2} - \alpha_2\right)t = \frac{1}{\sigma}\ln\left(\frac{\theta_I}{\theta_{E_2}}\right)\right\}$$

which is the first passage time of a Brownian motion with drift $\frac{1}{\sigma}\left(\frac{\sigma^2}{2} - \alpha_2\right)$ throught a level $\frac{1}{\sigma}\ln\left(\frac{\theta_I}{\theta_{E_2}}\right)$.

If $\frac{\sigma^2}{2} - \alpha_2 > 0$, as $\theta_I > \theta_{E_2}$ then the expected value is finite and the result follows in view of Proposition 8.5 of ?. Otherwise it is infinite, as the state $\frac{1}{\sigma}\ln\left(\frac{\theta_I}{\theta_{E_2}}\right)$ is null-recurrent. □

B.12 Proof of Proposition 12

The result follows in view of the example of section 10.9 of ?, with A given by α_1 and B given by σ^2, as the region Ω in our case is just the interval $[\theta_{E_1}, \theta_I]$ (a time homogeneous region). □

A Additional Model Details

A.1 Market 2

In order to derive the constant parameters (D_1, D_2, D_3 and G_1) of the value function and the exit thresholds for the two cases, we study the decision to exit the market. Denoting the exit threshold by θ_{E2}, we can write down the following value matching and smooth pasting conditions for the case $\theta_{E_2} < \theta_S$:

$$
\begin{aligned}
V_{2,2}(\theta)|_{\theta=\theta_{E_2}} &= 0, && (75)\\
\left.\frac{\partial V_{2,2}(\theta_2)}{\partial \theta}\right|_{\theta=\theta_{E_2}} &= 0, && (76)\\
V_{2,1}(\theta)|_{\theta=\frac{K_2}{\gamma}} &= V_{2,2}(\theta)|_{\theta=\frac{K_2}{\gamma}}, && (77)\\
\left.\frac{\partial V_{2,1}(\theta)}{\partial \theta}\right|_{\theta=\frac{K_2}{\gamma}} &= \left.\frac{\partial V_{2,2}(\theta)}{\partial \theta}\right|_{\theta=\frac{K_2}{\gamma}}, && (78)
\end{aligned}
$$

where

$$
V_2(\theta, K_2) = \begin{cases} \frac{\gamma K_2}{r-\alpha_2}\theta - \frac{K_2^2+F}{r} + D_1\theta^{\beta_4} & \text{for } \theta > \frac{K_2}{\gamma} \quad =: V_{2,1}(\theta), \\ -\frac{F}{r} + D_2\theta^{\beta_3} + D_3\theta^{\beta_4} & \text{for } \theta \leq \frac{K_2}{\gamma} \quad =: V_{2,2}(\theta)). \end{cases} \quad (79)
$$

Equations (77) and (78) account for the fact that the value function has to be smooth in $\theta = \frac{K_2}{\gamma}$. Solving equations (75) to (78) one can easily derive the exit threshold θ_{E_2} and the expressions for the parameters D_1, D_2 and D_3:

$$
\begin{aligned}
\theta_{E_2} &= \left(\frac{\beta_4\alpha_2 - r}{\beta_4(r-\alpha_2)}\right)^{\frac{-1}{\beta_3}} F^{\frac{1}{\beta_3}} \frac{1}{\gamma} K_2^{1-\frac{2}{\beta_3}},\\
D_1 &= \frac{r-\beta_3\alpha_2}{r(r-\alpha_2)(\beta_3-\beta_4)}\gamma^{\beta_4}K_2^{2-\beta_4} + D_3,\\
D_2 &= \frac{r-\beta_4\alpha_2}{r(r-\alpha_2)(\beta_3-\beta_4)}\gamma^{\beta_3}K_2^{2-\beta_3},\\
D_3 &= \theta_{E_2}^{-\beta_4}\frac{\beta_3}{\beta_3-\beta_4}\frac{F}{r}.
\end{aligned}
$$

For the second case, i.e. $\theta_{E_2} > \theta_S$, the value matching and smooth pasting conditions are given by

$$
\begin{aligned}
V_2(\theta)|_{\theta=\theta_{E_2}} &= 0, && (80)\\
\left.\frac{\partial V_2(\theta)}{\partial \theta}\right|_{\theta=\theta_{E_2}} &= 0. && (81)
\end{aligned}
$$

Solving this equation system one obtains

$$
\begin{aligned}
\theta_{E_2} &= \frac{r-\alpha_2}{r\gamma}\frac{\beta_4}{\beta_4-1}\left(K_2 + \frac{F}{K_2}\right),\\
G_1 &= -\frac{\gamma K_2}{r-\alpha_2}\frac{1}{\beta_4}\left(\frac{r-\alpha_2}{r\gamma}\frac{\beta_4}{\beta_4-1}\left(K_2+\frac{F}{K_2}\right)\right)^{1-\beta_4}.
\end{aligned}
$$

A.2 Market 1

In order to derive the constant parameters and the investment as well as the exit threshold (θ_I and θ_{E_1}) we use the value matching and smooth pasting conditions. Those are given for the three different cases in the following.

A.2.1 Case 1 - $\theta_{E_1} > \theta_{S_1}$ and $\theta_{E_2} \leq \theta_{S_2}$

The value matching and smooth pasting conditions for case 1 are given by:

$$\begin{aligned}
V_1(\theta)|_{\theta=\theta_{E_1}} &= 0 && (82)\\
\left.\frac{\partial V_1(\theta)}{\partial \theta}\right|_{\theta=\theta_{E_1}} &= 0 && (83)\\
V_1(\theta)|_{\theta=\theta_I} &= V_2(\theta)|_{\theta=\theta_I} - \delta K_2 && (84)\\
\left.\frac{\partial V_1(\theta)}{\partial \theta}\right|_{\theta=\theta_I} &= \left.\frac{\partial V_2(\theta)}{\partial \theta}\right|_{\theta=\theta_I} && (85)
\end{aligned}$$

Equations (82) and (83) lead to the following expressions for the parameters A_1 and A_2:

$$\begin{aligned}
A_1 &= \frac{a_1(\beta_2 - 1)\theta_{E_1}^{1-\beta_1} - a_2\beta_2\theta_{E_1}^{-\beta_1}}{\beta_1 - \beta_2}\\
A_2 &= \frac{a_1(\beta_1 - 1)\theta_{E_1}^{1-\beta_2} - a_2\beta_1\theta_{E_1}^{-\beta_2}}{\beta_2 - \beta_1}
\end{aligned}$$

From equations (84) and (85) (alongside the first two) we obtain equations (8) and (9) which implicity give the investment and exit threshold θ_I and θ_{E_1}.

A.2.2 Case 2 - $\theta_{E_1} < \theta_{S_1} < \theta_I$ and $\theta_{E_2} \leq \theta_{S_2}$

For case 2 the corresponding value matching and smooth pasting conditions are given by:

$$\begin{aligned}
V_{1,2}(\theta)|_{\theta=\theta_{E_1}} &= 0 && (86)\\
\left.\frac{\partial V_{1,2}(\theta)}{\partial \theta}\right|_{\theta=\theta_{E_1}} &= 0 && (87)\\
V_{1,1}(\theta)|_{\theta=\frac{c+K_1}{\mu}} &= V_{1,2}(\theta)|_{\theta=\frac{c+K_1}{\mu}} && (88)\\
\left.\frac{\partial V_{1,1}(\theta)}{\partial \theta}\right|_{\theta=\frac{c+K_1}{\mu}} &= \left.\frac{\partial V_{1,2}(\theta)}{\partial \theta}\right|_{\theta=\frac{c+K_1}{\mu}} && (89)\\
V_{1,1}(\theta)|_{\theta=\theta_I} &= V_2(\theta)|_{\theta=\theta_I} - \delta K_2 && (90)\\
\left.\frac{\partial V_{1,1}(\theta)}{\partial \theta}\right|_{\theta=\theta_I} &= \left.\frac{\partial V_2(\theta)}{\partial \theta}\right|_{\theta=\theta_I} && (91)
\end{aligned}$$

Equations (86) through (89) leads to the following expressions for the parameters $B_1 - B_4$:

$$\begin{aligned}
B_1 &= \frac{-a_1(\beta_2 - 1)\left(\frac{c+K_1}{\mu}\right)^{1-\beta_1} + \beta_2\frac{cK_1+K_1^2}{r}\left(\frac{c+K_1}{\mu}\right)^{-\beta_1}}{\beta_2 - \beta_1} + B_3\\
B_2 &= \frac{-a_1(\beta_1 - 1)\left(\frac{c+K_1}{\mu}\right)^{1-\beta_2} + \beta_1\frac{cK_1+K_1^2}{r}\left(\frac{c+K_1}{\mu}\right)^{-\beta_2}}{\beta_1 - \beta_2} + B_4\\
B_3 &= \theta_{E_1}^{-\beta_1}\frac{\beta_2}{\beta_2 - \beta_1}\frac{F}{r}\\
B_4 &= \theta_{E_1}^{-\beta_2}\frac{\beta_1}{\beta_1 - \beta_2}\frac{F}{r}
\end{aligned}$$

The equations (10) and (11) are obtained from (90) and (91), where the first explicitly gives an expression for θ_{E_1} and the latter an implicit expression for θ_I.

A.2.3 Case 3 - $\theta_{E_1} < \theta_I < \theta_{S_1}$ and $\theta_{E_2} \leq \theta_{S_2}$

The value matching and smooth pasting conditions for this case are given by

$$V_1(\theta)|_{\theta=\theta_{E_1}} = 0 \tag{92}$$
$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_1}} = 0 \tag{93}$$
$$V_1(\theta)|_{\theta=\theta_I} = V_2(\theta)|_{\theta=\theta_I} - \delta K_2 \tag{94}$$
$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_I} = \left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_I} \tag{95}$$

As before, from the first two equations (92) and (93) result in the following expressions for the parameters C_1 and C_2, respectively.

$$C_1 = \theta_{E_1}^{-\beta_1}\frac{\beta_2}{\beta_2-\beta_1}\frac{F}{r}$$
$$C_2 = \theta_{E_1}^{-\beta_2}\frac{\beta_1}{\beta_1-\beta_2}\frac{F}{r}$$

The equations (12) and (13) are obtained from (94) and (95), where the first explicitly gives an expression for θ_{E_1} and the latter an implicit expression for θ_I.

For the other three cases we have the same value matchings and smooth pastings, the only change is on V_2, therefore the values of A, B and C do not change. Whereas the expressions for θ_I and θ_{E_1} change accordingly, but are obtained like in the previous cases.

A.3 Market 2

In order to derive the constant parameter A we study the decision to exit the market, we can write down the following value matching and smooth pasting conditions:

$$V_2(\theta) = \frac{\gamma^2\theta^2}{4(r-2\alpha_2-\sigma^2)} - \frac{F}{r} + A\theta^{\beta_4}.$$

$$V_2(\theta)|_{\theta=\theta_{E_2}} = 0, \tag{96}$$
$$\left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_2}} = 0. \tag{97}$$

Solving equations (96) to (97) one can easily derive the exit threshold θ_{E_2} and the expressions for the parameter A:

$$\begin{aligned} \theta_{E_2} &= \sqrt{\frac{F}{\gamma^2 r} 4(r - 2\alpha_2 - \sigma^2)\left(\frac{\beta_4}{\beta_4 - 2}\right)} \\ A &= \theta_{E_2}^{-\beta_4} \frac{F}{r}\left(\frac{2}{2 - \beta_4}\right). \end{aligned}$$

A.4 Market 1

A.4.1 Case 1 - $\theta_{E_1} > \frac{c}{\mu}$

Value matching and smooth pasting for this case give:

$$V_1(\theta)|_{\theta=\theta_{E_1}} = 0 \tag{98}$$

$$\left.\frac{\partial V_1(\theta)}{\partial \theta}\right|_{\theta=\theta_{E_1}} = 0 \tag{99}$$

$$V_1(\theta)|_{\theta=\theta_I} = V_2(\theta)|_{\theta=\theta_I} - I \tag{100}$$

$$\left.\frac{\partial V_1(\theta)}{\partial \theta}\right|_{\theta=\theta_I} = \left.\frac{\partial V_2(\theta)}{\partial \theta}\right|_{\theta=\theta_I} \tag{101}$$

Equations (98) and (99) lead to the following expressions for the parameters A_1 and A_2:

$$\begin{aligned} A_1 &= \frac{b_1(\beta_2 - 2)\theta_{E_1}^{2-\beta_1} - b_2(\beta_2 - 1)\theta_{E_1}^{1-\beta_1} + b_3\beta_2\theta_{E_1}^{-\beta_1}}{\beta_1 - \beta_2} \\ A_2 &= \frac{b_1(\beta_1 - 2)\theta_{E_1}^{2-\beta_2} - b_2(\beta_1 - 1)\theta_{E_1}^{1-\beta_2} + b_3\beta_1\theta_{E_1}^{-\beta_2}}{\beta_2 - \beta_1} \end{aligned}$$

From equations (100) and (101) (alongside the first two) we obtain equations (23) and (24) which implicity give the investment and exit threshold θ_I and θ_{E_1}.

A.4.2 Case 2 - $\theta_{E_1} < \frac{c}{\mu} < \theta_I$

Value matching and smooth pasting for this case give:

$$V_{1,2}(\theta)|_{\theta=\theta_{E_1}} = 0 \tag{102}$$

$$\left.\frac{\partial V_{1,2}(\theta)}{\partial \theta}\right|_{\theta=\theta_{E_1}} = 0 \tag{103}$$

$$V_{1,1}(\theta)|_{\theta=\frac{c}{\mu}} = V_{1,2}(\theta)|_{\theta=\frac{c}{\mu}} \tag{104}$$

$$\left.\frac{\partial V_{1,1}(\theta)}{\partial \theta}\right|_{\theta=\frac{c}{\mu}} = \left.\frac{\partial V_{1,2}(\theta)}{\partial \theta}\right|_{\theta=\frac{c}{\mu}} \tag{105}$$

$$V_{1,1}(\theta)|_{\theta=\theta_I} = V_2(\theta)|_{\theta=\theta_I} - I \tag{106}$$

$$\left.\frac{\partial V_{1,1}(\theta)}{\partial \theta}\right|_{\theta=\theta_I} = \left.\frac{\partial V_2(\theta)}{\partial \theta}\right|_{\theta=\theta_I} \tag{107}$$

Equations (102) through (105) leads to the following expressions for the parameters $B_1 - B_4$:

$$\begin{aligned}
B_1 &= \frac{b_1(\beta_2-2)\left(\frac{c}{\mu}\right)^{2-\beta_1} - b_2(\beta_2-1)\left(\frac{c}{\mu}\right)^{1-\beta_1} + \beta_2(b_3+b_5)\left(\frac{c}{\mu}\right)^{-\beta_1}}{\beta_1-\beta_2} + B_3 \\
B_2 &= \frac{b_1(\beta_1-2)\left(\frac{c}{\mu}\right)^{2-\beta_2} - b_2(\beta_1-1)\left(\frac{c}{\mu}\right)^{1-\beta_2} + \beta_1(b_3+b_5)\left(\frac{c}{\mu}\right)^{-\beta_2}}{\beta_2-\beta_1} + B_4 \\
B_3 &= \theta_{E_1}^{-\beta_1}\frac{\beta_2}{\beta_2-\beta_1}b_5 \\
B_4 &= \theta_{E_1}^{-\beta_2}\frac{\beta_1}{\beta_1-\beta_2}b_5
\end{aligned}$$

The equations (25) and (26) are obtained from (106) and (107), where the first explicitly gives an expression for θ_{E_1} and the latter an implicit expression for θ_I.

A.4.3 Case 3 - $\theta_{E_1} < \theta_I < \frac{c}{\mu}$

The value matching and smooth pasting conditions for this case are given by

$$V_1(\theta)|_{\theta=\theta_{E_1}} = 0 \tag{108}$$

$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_{E_1}} = 0 \tag{109}$$

$$V_1(\theta)|_{\theta=\theta_I} = V_2(\theta)|_{\theta=\theta_I} - I \tag{110}$$

$$\left.\frac{\partial V_1(\theta)}{\partial\theta}\right|_{\theta=\theta_I} = \left.\frac{\partial V_2(\theta)}{\partial\theta}\right|_{\theta=\theta_I} \tag{111}$$

As before, from the first two equations (92) and (93) result in the following expressions for the parameters C_1 and C_2, respectively.

$$C_1 = \theta_{E_1}^{-\beta_1}\frac{\beta_2}{\beta_2-\beta_1}b_5$$
$$C_2 = \theta_{E_1}^{-\beta_2}\frac{\beta_1}{\beta_1-\beta_2}b_5$$

The equations (27) and (28) are obtained from (110) and (111), where the first explicitly gives an expression for θ_{E_1} and the latter an implicit expression for θ_I.

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