CHARACTERIZATIONS OF STABILITY

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Motive for Equilibrium Selection

The original Nash definition allows

1. Multiple equilibria
2. Dominated strategies to be used
3. Implausible beliefs in extensive form
4. Unstable equilibria that disappear
   if the game is perturbed

Selection tries to exclude 2-3-4

Some Normal-Form Selections

■ Perfect	*Selten*	*1975*
■ Proper	*Myerson*	*1978*
■ Lexicographic	*Blume-Brandenberger-Dekel*	*1991*
■ Stable sets	*Kohlberg-Mertens*	*1986*
	Mertens	*1989*

Some Extensive-Form Selections

- Subgame Perfect *Selten 1965*
- Extensive-form Perfect *Selten 1975*
- Sequential *Kreps-Wilson 1982*
- Quasi-Perfect *van Damme 1984*

  - ***A basic goal is to <u>unify</u> the Normal-Form and Extensive-Form perspectives***

The Kohlberg-Mertens Program

1. Specify desirable properties or axioms
   - Assume <u>set</u>-valued selections
     - Genericity of extensive game $\Rightarrow$ (within a component) all equilibria have the same outcome
2. Define selections that achieve basic criteria
   - Invariance, admissibility, backward & forward induction …

- Mertens-Stability meets ALL their criteria *<u>but</u>* it depends on a topological construction
- We report results about
  - Hyperstability
  - Stability

♦ Hyperstable Set of Equilibria

- *Definition*: Each payoff perturbation of each inflation of the game has an equilibrium whose deflation is near the set
  - Inflation appends redundant pure strategies
    - Treats some mixed strategies as pure strategies
  - Deflation converts back to equivalent mixture of the original pure strategies
- Inflation/Deflation invoke the Axiom of Invariance to presentation effects

Characterization of Hyperstable Components

- ***Theorem*: A component is hyperstable**
  if and only if **its index is nonzero**
  – Thus hyperstability is a topological property
  Verify hyperstability by computing an index

<u>Relation to prior literature</u>:

- *Definition*: A component of fixed points is <u>essential</u>
  if each map nearby has a fixed point nearby
- *Theorem*: A component of fixed points is essential
  iff its index is nonzero *[O'Neill 1953]*

So hyperstable components are essential
whereas Mertens 1989 imposes essentiality

Main Steps of Proof

To show Index = 0 $\Rightarrow$ not-hyperstable

1. Index = 0 $\Rightarrow$ $\exists$ map $\sigma \to G(\sigma) = G \oplus g(\sigma)$ to nearby perturbed games such that no σ near the component is an equilibrium of $G(\sigma)$
   - This step extends KM's Structure Theorem
2. Using simplicial approximation of map g construct perturbed inflated games $G^*(\sigma)$
3. Hyperstability $\Rightarrow$ $(\exists\sigma^*)$ σ^* is an equilibrium of $G^*(\sigma)$, where σ = deflation of σ^*
   $\Rightarrow$ σ is an equilibrium of $G(\sigma)$
   $\Rightarrow$ contradiction !

♦ Stable Set of Equilibria

- *Definition*: Each perturbed game obtained by <u>shrinking</u> the simplex of mixed strategies has an equilibrium near the set
  - <u>Shrinking</u> via η means each $\sigma \to (1-\varepsilon)\sigma + \varepsilon\eta$
  - KM require a *minimal* stable set
- A stable set is <u>truly</u> perfect
  - It is perfect against every tremble η
- Stability excludes dominated strategies
  - But hyperstability allows them

Stability Characterization
for 2 players

Theorem: A closed set S contains a KM-stable set *if and only if*

- For each tremble η there exist profiles σ & τ and $\varepsilon \in (0,1]$, where $\sigma \in$ S, such that each pure strategy used in <u>either</u> σ *or* τ is an optimal reply to <u>both</u> σ *and* $(1-\varepsilon)\tau + \varepsilon\eta$
  - That is, perturbing τ by tremble η "respects preferences" *[Blume-Brandenberger-Dekel 1991]*
  - Generalizes the characterization for generic signaling games *[Cho-Kreps & Banks-Sobel 1988]*
- ***N players***: analog lexicographic condition

Axioms for Stability

Our approach mixes ***normal-form*** and ***extensive-form*** criteria

- ♦ Our normal-form criterion is

- **Axiom 1** ***Weak Invariance***
  Selection should be immune to inflation
  - – That is, exclude presentation effects

- ♦ Our extensive-form criterion is

- **Axiom 2** ***Strong Backward Induction***
  For an extensive game, trembles should select admissible sequential equilibria
  - – Formulation of Axiom 2 uses ε-Quasi-Perfection

ε-Quasi-Perfect in Extensive Game with Perfect Recall

- *Definition*: An action at an information set is <u>optimal</u> (for tremble η) if it begins an optimal continuation strategy using beliefs induced by perturbations toward η
- *Definition*: $\sigma > 0$ is <u>ε-QP</u> if <u>sub</u>optimal actions have cond. probabilities $\leq \varepsilon$ *[van Damme 1984]*
- Proper ⇒ QP ⇒ Sequential equilibrium
  <u>*but*</u> QP excludes conditionally dominated strategies

Axiom 2 Using Strong Quasi-Perfect

Require lower bounds on behavior strategies
Then:

- **Axiom 2**: Each tremble should select some strong-QP-equilibrium from the set
  - – Axiom 2 is stringent: robust to <u>all</u> trembles
    Requires that selection is "truly" Quasi-Perfect
  - – Use of trembles is akin to Mertens' use of a "germ" inducing beliefs
    - Mertens-stable sets satisfy Axioms 1 and 2
    - Could use lexicographic or equiproper instead?

Sufficiency Theorem

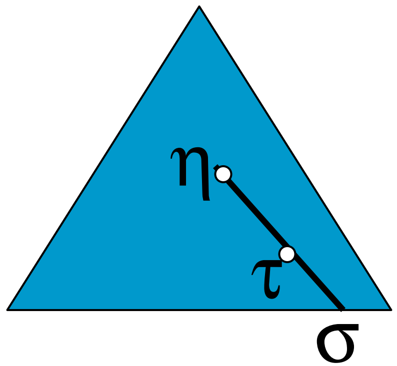
- **Theorem**: Axioms 1 & 2 imply that a selected set includes a KM-stable set
  - Corollary: If an extensive game is generic then selection yields a stable outcome
  - Axiom *Proper* is sufficient for simple games (generic signaling, outside-option, and perfect-information games) but insufficient generally
  - Add Axiom *Homotopy* $\Rightarrow$ Mertens-stable set

Sketch of Proof

1. Construct inflated extensive game in which each player chooses either
   - the tremble with minimum probability $\geq \varepsilon$ or
   - plays the original game with the maximum probability of a suboptimal strategy $\leq \varepsilon \times \varepsilon$
2. Each tremble and ε-QP sequence induces a lexicographic equilibrium
3. The lexicographic equilibrium satisfies the Characterization Theorem for KM-stable set

Summary Remarks

- **Hyperstable** component $\Leftrightarrow$ <u>essential</u> set
  - So select <u>within</u> a hyperstable component verified by computing its index
  - This is implicit in Mertens' construction
- **Stability** $\Leftarrow$ Invariance + (in extensive game) <u>conditionally admissible</u> strategies
  - Here, formulated via Quasi-Perfect
- In both cases, <u>Invariance</u> is the key tool !
  - Enables mixing normal-form and extensive-form criteria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