# CHARACTERIZATION OF HYPERSTABILITY

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Abstract. A component of the equilibria of a finite game is hyperstable if and only if its index is nonzero.

## 1. Introduction

A cornerstone of the theory of games is the concept of equilibrium proposed by Nash (1950). Nash's definition requires that each player's strategy should be an optimal reply to other players' strategies. This minimal requirement does not generate a complete theory of rational play since most games have multiple equilibria. However, for applications in economics and other social sciences, some equilibria seem to be more plausible predictors of behavior than others. Selten (1965, 1975) initiated the study of equilibrium refinements; i.e., auxiliary criteria that select subsets of a game's equilibria with stronger properties. Hillas and Kohlberg (2002) survey the main proposals for strengthening Nash's definition by imposing additional criteria.

One of the most stringent criteria is hyperstability, proposed by Kohlberg and Mertens (1986) [KM hereafter]. Hyperstability requires a kind of continuity with respect to perturbations of players' payoffs in strategically equivalent games. Continuity is evidently necessary in applications—it would be unrealistic to suppose that decisions depend on infinitely precise specifications of payoffs, or in empirical studies, that structural estimation of an econometric model of interactions among firms could rely on fragile theoretical predictions. However, continuity is not the fundamental motivation. KM show that hyperstability and its weaker variants full stability and stability (which induce payoff perturbations by perturbing players' strategies) imply decision-theoretic principles considered desirable for an axiomatic characterization of rational behavior in strategic interactions.

Hyperstability invokes two such principles. The first, called Invariance, requires that equilibrium selection should be immune to treating a mixed strategy as an additional pure strategy. As an axiom, Invariance rules out presentation effects by requiring that equivalent equilibria are selected in equivalent games. The second, called Stability, requires that every nearby game should have a nearby equilibrium. Stability is invoked to obtain properties such as consistency with the results of iterative elimination of dominated strategies, and backward induction in games in extensive form. In particular, optimal strategies for continuation from an information set off the path of equilibrium play requires some theory of how such a contingency could occur. Consideration of perturbed payoffs or strategies provides one class of such theories, and other theories, such

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http://faculty-gsb.stanford.edu/wilson/pdf%20files/Hyperstability040701.pdf.

as sequential equilibrium and lexicographic equilibrium (Blume, Brandenberger, and Dekel, 1991), can be cast in terms of perturbations either as limits or by using techniques of nonstandard analysis.

The principles of Invariance and Stability are applied as follows to obtain the definition of hyperstability.

- **Equivalence.** The relation of equivalence between games is defined as follows. First, say that two strategies of one player are equivalent if for every profile of other players' strategies they yield the same payoff for every player. A pure strategy of a player is redundant if that player has another pure or mixed strategy that is equivalent. From a game $G$ one obtains its reduced form $G_*$ by deleting redundant pure strategies until none remain. The reduced form is unique apart from the payoff-irrelevant names of the remaining pure strategies. Two games $G$ and $G'$ are equivalent if their reduced forms are the same; viz., $G_* = G'_*$. If $\sigma$ is a profile of players' strategies in a game $G$ then its reduced form $\sigma_*$ is the profile of equivalent strategies in the reduced form $G_*$.
- **Hyperstability.** A subset of equilibria in reduced form is hyperstable if every sufficiently small perturbation of every equivalent game has an equilibrium whose reduced form is arbitrarily near the subset. That is, a subset $E$ of the equilibria of a game $G$ is hyperstable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every perturbation $G''$ of a game $G'$ equivalent to $G$ for which $\|G'' - G'\| \leqslant \delta$ has an equilibrium $\sigma''$ such that $\|\sigma''_* - \sigma_*\| \leqslant \varepsilon$ for some equilibrium $\sigma \in E$.[1]

Of special importance are the subsets that are components; i.e., maximal connected sets of equilibria. Kohlberg and Mertens (1986, Proposition 1) prove that for every game its equilibria are divided into a finite number of components, of which at least one is hyperstable.

From the definition of hyperstability one might suppose that verifying whether a component is hyperstable is a formidable challenge. However, our purpose here is to establish that hyperstability is a purely topological property. Moreover, there is a computable topological invariant, called the index of the component, that precisely characterizes whether a component is hyperstable. Recall that an equilibrium of a game can be characterized as a fixed point of a map from the space of mixed-strategy profiles into itself; for example, Nash (1951) and Gül, Pearce, and Stacchetti (1993) specify two such maps. In the theory of algebraic topology a set of axioms characterizes an assignment of an integer to each component of the fixed points of maps (Dold, 1972, VII.5). This integer is called the index of the component. It is known, moreover, that the indices of the components of the equilibria of a game are independent of the map used to characterize equilibria (DeMichelis and Germano, 2000; Govindan and Wilson, 1998); indeed, in Section A.1 of the Appendix we define an index that depends only on the best-reply correspondence, which is intrinsic to each game regardless of which map might be used to characterize equilibria as fixed points. For every game the sum of the indices of the components of its equilibria is $+1$ (Gül, Pearce, and Stacchetti, 1993; Kohlberg and Mertens, 1986, Theorem 1). The index is invariant under addition or deletion of redundant strategies (Govindan and Wilson, 1997, and Theorem A.3 here); in particular, the index of a component $C$ of the equilibria of a game $G$ is the same as the index of the component $C_*$ in equivalent strategies of the game's reduced form $G_*$.

The index of a component can be computed as the sum of the indices of the nearby equilibria of any sufficiently nearby game. If this nearby game is generic then each equilibrium is a singleton component and its index can be computed as the sign ($+1$ or $-1$) of the determinant of a Jacobian matrix. Gül, Pearce, and

[1]We use $\ell_\infty$ norm throughout.

Stacchetti (1993) describe one formulation and alternatives are described in Govindan and Wilson (2002a) for games in normal form and in Govindan and Wilson (2002b) for games in extensive form.

Our main theorem is the following generalization of a result established by von Schemde (2004) for 2-player outside-option games:

**Theorem 1.1.** *A component of the equilibria of a game is hyperstable if and only if its index is nonzero.*

Thus the equilibrium refinement that selects the hyperstable components is characterized by the property that it selects those components whose indices are nonzero. Even though the definition of hyperstability considers every perturbation of every equivalent game, one verifies hyperstability of a component simply by computing its index and verifying that it is nonzero.

The restriction to components of equilibria is immaterial for games derived from an extensive form with perfect recall and generic payoffs since for such games all equilibria in a component induce the same probability distribution over outcomes; i.e., they differ only off the path of equilibrium play (Kreps and Wilson, 1982; Govindan and Wilson, 2001a). One can derive subsets of a hyperstable component with stronger properties; e.g., to ensure that the equilibria selected use only admissible pure strategies, KM focus on minimal closed subsets that are either fully stable or stable depending on the class of strategy perturbations considered. Especially useful is the series of facts established by KM: (a) strategy perturbations induce payoff perturbations so a hyperstable component necessarily contains subsets that are fully stable and stable; (b) a fully stable subset contains a proper equilibrium as defined by Myerson (1978); (c) a proper equilibrium induces a sequential equilibrium, as defined by Kreps and Wilson (1982), in every extensive form with that normal form, and thus implements the principle of backward induction. Alternatively, Govindan and Wilson (2004) show that two axioms imply that a selected set of equilibria should contain a stable subset: (1) invariance and (2) the requirement that each perturbation of players' strategies should induce a further refinement by selecting among the quasi-perfect equilibria, as defined by van Damme (1984). Like a proper equilibrium, a quasi-perfect equilibrium of an extensive-form game induces a sequential equilibrium that does not use weakly dominated strategies. All these desirable properties are implied by the revised (and seemingly much stronger) definition of stability proposed by Mertens (1989). Mertens invokes as part of his definition a stringent topological property akin to the conclusion of Theorem 1.1. One interpretation of our result is that it is much simpler and vastly easier computationally to select a hyperstable component, and then from that component select a subset or a single equilibrium with additional desirable properties, as advocated by Wilson (1997). Another approach, fully implemented only for generic extensive-form games with perfect information, is exemplified by Aumann's (1995) result that common knowledge of rationality implies that the outcome of a game is the one predicted by subgame perfection, as defined by Selten (1965); namely, the unique outcome of all the equilibria in the only component that is hyperstable.

Section 2 establishes notation. For the general reader, the main ideas in the proof are outlined in Section 3 for the case of 2 players. Theorem 1.1 for the general case is proved in Section 4. In Section 5 we comment on the implications of our result for Mertens' stronger definition of stability. Appendices A and B provide tools used in the text. Appendix C reports a numerical example.

## 2. Formulation

We consider games with a finite set $N$ of players, $|N| \geqslant 2$. Each player $n \in N$ has a finite set $S_n$ of pure strategies. Interpret a pure strategy $s_n$ as a vertex of player $n$'s simplex $\Sigma_n = \Delta(S_n)$ of mixed strategies. The sets of profiles of pure and mixed strategies are $S = \prod_n S_n$ and $\Sigma = \prod_n \Sigma_n$. When focusing on a player $n$, we use $S_{-n} = \prod_{m \neq n} S_m$ and $\Sigma_{-n} = \prod_{m \neq n} \Sigma_m$ to denote profiles of the other players' pure and mixed strategies.

Given $N$ and $S$, each game $G$ is described by its payoff function $\hat{G} : S \to \mathbb{R}^N$ from profiles of pure strategies to payoffs for each of the players. Thus a game is specified by a point in the Euclidean space $\mathbb{R}^{S \times N}$. We use $G$ to denote the multilinear extension of $\hat{G}$ from profiles of mixed strategies to players' expected payoffs. In particular, player $n$'s expected payoffs from his pure strategies are specified by the map $G_n : \Sigma_{-n} \to \mathbb{R}^{S_n}$, where

$$G_{ns}(\sigma) = \sum_{t \in S_{-n}} \hat{G}_n(s,t) \prod_{m \neq n} \sigma_m(t_m)$$

for each pure strategy $s \in S_n$ and profile $\sigma \in \Sigma$. Note that $G_n$ depends only on the profile $\sigma_{-n} \in \Sigma_{-n}$ of other players' mixed strategies. The corresponding best-reply correspondence is

$$\mathrm{BR}(\sigma^0) = \{\sigma \in \Sigma \mid (\forall\ n)(\forall\ \tau_n \in \Sigma_n)\ \sigma_n' G_n(\sigma^0_{-n}) \geqslant \tau_n' G_n(\sigma^0_{-n})\}\,.$$

A profile $\sigma \in \Sigma$ of mixed strategies is an equilibrium of the game $G$ if each player's strategy $\sigma_n$ is an optimal reply to the other players' strategies; that is, $[\tau_n - \sigma_n]' G_n(\sigma) \leqslant 0$ for all $\tau_n \in \Sigma_n$. An equilibrium component is a maximal connected set of equilibria, and thus compact. Equilibria can be characterized as fixed points of a map $\Phi : \Sigma \to \Sigma$ as follows (Gül, Pearce, and Stacchetti, 1993). Let $r_n : \mathbb{R}^{S_n} \to \Sigma_n$ be the piecewise-affine function that is the retraction mapping each point in $\mathbb{R}^{S_n}$ to the point of $\Sigma_n$ that is nearest in Euclidean distance; i.e., $r_n(z_n)$ is the unique solution $r \in \Sigma_n$ to the variational inequality $[\tau_n - r]'[z_n - r] \leqslant 0$ for all $\tau_n \in \Sigma_n$ (actually, it suffices to consider only the finite number of pure strategies $\tau_n \in S_n$). Let $\mathcal{Z} = \prod_n \mathbb{R}^{S_n}$ and define $r : \mathcal{Z} \to \Sigma$ via $r(z)_n = r_n(z_n)$ for each player $n$. Also define $w : \Sigma \to \mathcal{Z}$ via $w(\sigma) = \sigma + G(\sigma)$. Then the above definition translates to the alternative definition that $\sigma$ is an equilibrium if and only if $\sigma = [r \circ w](\sigma)$. Hence the equilibria of $G$ are precisely the fixed points of the map $\Phi \equiv r \circ w : \Sigma \to \mathcal{Z} \to \Sigma$. Its commuted version is $F \equiv w \circ r : \mathcal{Z} \to \Sigma \to \mathcal{Z}$. The sets of fixed points of $\Phi$ and $F$ are homeomorphic: $\sigma = r(z)$ is a fixed point of $\Phi$ iff $z = w(\sigma)$ is a fixed point of $F$. In this formulation, the topological *index* of a component $C$ is the integer that is the local degree of $\Phi$'s displacement map $\varphi(\sigma) \equiv \sigma - \Phi(\sigma)$, restricted to any small neighborhood of $C$ disjoint from other components. Similarly, $D = w(C)$ is the corresponding component of fixed points of $F$ and its index is the local degree of $F$'s displacement map $f : \mathcal{Z} \to \mathcal{Z}$, $f(z) = z - F(z)$. The indices of $C$ and $D$ are the same (Dold, 1972, VII.5.9).

A restricted class of payoff perturbations of a game $G$ perturbs each player's payoffs from his pure strategies independently of other players' behaviors. For each $g \in \mathcal{Z}$ define the perturbed game $G \oplus g$ by $(G \oplus g)_n(\sigma) = G_n(\sigma) + g_n$. Let $\mathcal{E}_G = \{(g, \sigma) \in \mathcal{Z} \times \Sigma \mid \sigma$ is an equilibrium of $G \oplus g\}$ be the graph of equilibria over this class of perturbations. KM's Theorem 1 implies that the map $\theta : \mathcal{E}_G \to \mathcal{Z}$, $\theta(g, \sigma) = \sigma + G(\sigma) + g$, is a homeomorphism; in particular, $\theta^{-1}(z) = (f(z), r(z))$. Consequently, $f = p_1 \circ \theta^{-1}$, where $p_1 : \mathcal{E}_G \to \mathcal{Z}$, $p_1(g, \sigma) = g$, is the projection to the first coordinate. Using an appropriate orientation of $\mathcal{E}_G$, the degree of

$\theta^{-1}$ is $+1$. Since the degree of a composition of maps is the product of their degrees, the local degree of $f$ is same as the local degree of the projection map $p_1$. Hence the index of a component $C$ of $G$ is the same as the degree of the projection map $p_1$ on any sufficiently small neighborhood of $(0, C)$ in the graph $\mathcal{E}_G$. As mentioned, Section A.1 presents an alternative definition of the index that depends only on the best-reply correspondence BR.

As described in Section 1, a profile $\sigma \in \Sigma$ for the game $G$ induces an equivalent profile $\sigma_* \in \Sigma_*$ of $G$'s reduced form $G_*$. Let $A_n$ be the matrix whose columns are the pure strategies in $S_n$ represented as mixed strategies in $\Sigma_{*n}$. Then $\sigma_{*n} = A_n\sigma_n$ and $G_n(\sigma) = A_n' G_{*n}(\sigma_*)$. A profile $\sigma \in \Sigma$ is an equilibrium of $G$ if and only if the equivalent profile $\sigma_* = (A_n\sigma_n)_{n\in N}$ is an equilibrium of $G_*$.

The properties of simplicial approximations used below are developed in Spanier (1966, Chapter 3) and in Section A.2 of the Appendix.

## 3. Sketch of the Proof for 2 Players

Because the proof of Theorem 1.1 in Section 4 is long and notationally complicated, in this section we sketch the key ideas for the special case of a component of the symmetric equilibria of a symmetric game with 2 players. (The properties of the index can be specialized to components of the symmetric equilibria of symmetric games.) Such a game is described by a single square matrix $G = (G_{ss'})$, where $G_{ss'}$ is the payoff to the "row" player who chooses the pure strategy $s$ when the "column" player chooses the pure strategy $s'$.

Interpreted for a symmetric game, the main complication in Theorem 4.2 involves establishing a generalization of the following fact. Suppose that the component $C_*$ of the symmetric equilibria of a game $G_*$ in reduced form has index zero, and let $U_*$ be a closed neighborhood of $C_*$ that contains no other equilibria of $G_*$. Then there exists a map $g$ from $\Sigma_*$ to an arbitrarily small neighborhood $V$ of the origin in $\mathbb{R}^{S_*}$ such that no strategy $\sigma_* \in U_*$ is a symmetric equilibrium of the perturbed game $G_* \oplus g(\sigma_*)$. Here we assume that such a map exists and illustrate the remainder of the proof that $C_*$ is not hyperstable. This requires construction of an equivalent symmetric game $G$ and a perturbation of $G$ that has no symmetric equilibrium whose reduced form is in $U_*$.

Without loss of generality, $g$ can be assumed to be a simplicial map using sufficiently fine simplicial subdivisions of $\Sigma_*$ and $V$ — that is, simplices in the domain are mapped linearly by $g$ to simplices in the range. As in Section B.2 of the Appendix, refine the simplicial subdivision of $\Sigma_*$ to obtain a polyhedral subdivision for which there is a convex function $\gamma : \Sigma_* \to \mathbb{R}$ with the property that $\gamma$ is linear on precisely the polyhedra of the subdivision. For the equivalent game $G$ let the set $S$ of pure strategies for each player be the set $P$ of vertices of this polyhedral subdivision, and let $A$ be the $|S_*| \times |P|$ matrix whose columns are the vertices in $P$ represented as points in $\Sigma_*$. Then $G = A'G_*A$ is the payoff matrix for a symmetric 2-player game whose reduced form is $G_*$. The component of its symmetric equilibria that reduce to ones in $C_*$ is $C = \{\sigma \in \Sigma \mid A\sigma \in C_*\}$ and similarly the corresponding neighborhood is $U = \{\sigma \in \Sigma \mid A\sigma \in U_*\}$. Define the matrix $B$ whose column indexed by $p \in P$ is $g(p)$. Then the game whose payoff matrix is $G^0 = A'G_*A + A'B$ is a perturbation of $G$. If the scalar $\alpha$ is sufficiently small then so too is the game $G^\alpha = A'[G_*A + B] \oplus \alpha c$, where $c_p = -\gamma(p)$ for each $p \in P$.

We claim that if $\alpha > 0$ then a symmetric equilibrium of $G^\alpha$ assigns positive probability only to vertices of a polyhedron in the subdivision. To see this, observe that for each strategy of the column player the row player's payoff in $G^0$ is the same for any two strategies $\sigma, \tau \in \Sigma$ with the same reduced form. Hence, in $G^\alpha$

with $\alpha > 0$, if $A\sigma = A\tau$ then $\sigma$ is an optimal reply for the row player only if $\sigma'[\alpha c] \geqslant \tau'[\alpha c]$. Thus, $\sigma$ is an optimal reply only if $\sigma \in \arg\min_{\tau\in\Sigma}\{\sum_{p\in P}\gamma(p)\tau(p) \mid A\tau = A\sigma\}$. Since $\gamma$ is convex, and linear on precisely the polyhedra of the subdivision, the support of any solution of this linear programming problem is a subset of the vertices of the polyhedron $P(\sigma_*)$ that contains the reduced form $\sigma_* = A\sigma$. In particular, if $\sigma$ is a symmetric equilibrium of $G^\alpha$ then its support is a subset of the vertices of $P(\sigma_*)$. Note further that $P(\sigma_*)$ is contained in the simplex $\Sigma(\sigma_*)$ of the simplicial subdivision of $\Sigma_*$ that contains $\sigma_*$; therefore the support of an equilibrium $\sigma$ is a subset of $\Sigma(\sigma_*)$. Since $g$ is linear on this simplex, $g(\sigma_*) = B\sigma$.

Now suppose to the contrary that $C_*$ is hyperstable. Then for each $\alpha$ in a sequence shrinking to zero each game $G^\alpha$ has a symmetric equilibrium $\sigma^\alpha$ whose reduced form $\sigma_*^\alpha = A\sigma^\alpha$ is in $U_*$ and for which $g(\sigma_*^\alpha) = B\sigma^\alpha$. Then the game $G^0$ has a symmetric equilibrium $\sigma$ that is the limit of a convergent subsequence, and it inherits the properties that $\sigma_* = A\sigma \in U_*$ and $g(\sigma_*) = B\sigma$. By definition of a symmetric equilibrium, $[\tau - \sigma]'A'[G_*A + B]\sigma \leqslant 0$ for all $\tau \in \Sigma$. Therefore $[\tau_* - \sigma_*]'[G_*\sigma_* + g(s_*)] \leqslant 0$ for all $\tau_* = A\tau \in \Sigma_*$. Thus $\sigma_* \in U_*$ is a symmetric equilibrium of the game $G_* \oplus g(\sigma_*)$. But this contradicts the fact about $g$ stated above. Thus, if $C_*$ has index zero then it is not hyperstable.

The proofs in Section 4 address the general case of asymmetric games with two or more players. The sketch above corresponds to Step 3 of the proof of Theorem 4.2, which follows the proof in Steps 1 and 2 that an analogous map $g$ exists when the component has index zero.

## 4. Proof of the Main Theorem

We now prove the two parts of Theorem 1.1 for the general case of games with an arbitrary number of players. Theorem 4.1 extends to the entire class of equivalent games the implication of nonzero index established by Ritzberger (1994).

**Theorem 4.1.** *An equilibrium component is hyperstable if its index is nonzero.*

*Proof.* It suffices to suppose that the game $G$ is in reduced form. Let $C$ be a component of the equilibria of $G$. Suppose that the index of $C$ is nonzero, say $d \neq 0$. We show that $C$ is hyperstable. Let $U$ be an open neighborhood of $C$ whose closure includes no other equilibria of $G$. Because no strategy in the boundary $\partial U$ is an equilibrium, $\bar\delta > 0$ where $\bar\delta = \min_{\sigma\in\partial U}[\max_{n\in N, s\in S_n} G_{n,s}(\sigma_{-n}) - \sigma_n' G_n(\sigma_{-n})]$. Let $G^*$ be a game whose reduced form is $G$, and let $C^*$ be the equilibrium component of $G^*$ whose reduced form is $C$. Let $\mathcal{E}^*$ be the graph of the equilibrium correspondence over the space of games with the same set of strategies as in $G^*$. Choose any $\delta \in (0, \bar\delta/2)$ and let $\mathbb{B}^*$ be the open ball around $G^*$ with radius $\delta$. Let $U^* \supset C^*$ be the set of mixed strategies of $G^*$ that reduce to strategies in $U$; note that a profile in $\partial U^*$ reduces to a profile in $\partial U$. Then $V^* = \mathcal{E}^* \cap (\mathbb{B}^* \times U^*)$ is a neighborhood of $(G^*, C^*)$ in the graph. Suppose $\sigma^* \in \partial U^*$ and let $\sigma$ be the corresponding profile in $\partial U$. Then there exists a pure strategy $s$ for some player $n$ whose payoff $\pi_s(\sigma)$ in $G$ from $s$ is greater than the payoff from the reduced form $\sigma_n$ of $\sigma_n^*$ by at least $\delta$. For a game $\hat G^* \in B^*$, the payoff from $s$ is strictly greater than $\pi_s(\sigma) - \bar\delta/2$ while the payoff from $\sigma_n^*$ is its strictly less than $\pi_s(\sigma) - \bar\delta/2$. Thus, $\sigma^*$ cannot be an equilibrium of $\hat G^*$. Therefore, $\hat G^*$ has no equilibrium in $\partial U^*$. Consequently, the projection map $P^* : V^* \to \mathbb{B}^*$ is proper: $(P^*)^{-1}$ maps compact subsets to compact sets. As shown in Section A.2 of the Appendix, the index and degree of $C$ and $C^*$ agree. Therefore, the local degree of $G^*$ under $P^*$ is $d$. Because $P^*$ is a proper map, this implies that the local degree of each game $\hat G^* \in \mathbb{B}^*$ is $d$ (Dold, 1972, VIII.4.5). Therefore, the sum of the indices of equilibrium components of $\hat G^*$ in

$U^*$ is $d$. Since $d \neq 0$, $\hat{G}^*$ has an equilibrium in $U^*$. Since $G^*$ could be any game whose reduced form is $G$, and every game in its neighborhood $\mathbb{B}^*$ has an equilibrium in $U^*$, $C$ is hyperstable. $\square$

**Theorem 4.2.** *An equilibrium component is hyperstable only if its index is nonzero.*

*Proof.* Let $C$ be a component of the equilibria of a game $G$. Assume that the index of $C$ is zero. Let $\varepsilon > 0$ be sufficiently small that the closed $\varepsilon$-neighborhood $U$ of $C$ in $\Sigma$ is disjoint from other components. We show that for every $\delta > 0$ there exists an equivalent game $\tilde{G}$ and a perturbation $\tilde{G}_\delta$ of $\tilde{G}$ such that $\|\tilde{G} - \tilde{G}_\delta\| \leqslant \delta$ and the perturbed game $\tilde{G}_\delta$ has no equilibrium equivalent to an equilibrium in $U$. Thus, fix some $\delta > 0$. The construction of the equivalent game $\tilde{G}$ with the requisite property is done in several steps.

For $\beta > 0$ say that a strategy $\tau_n$ of player $n$ is a $\beta$-reply against $\sigma \in \Sigma$ if $G_n(s, \sigma_{-n}) - G_n(\tau_n, \sigma_{-n}) \leqslant \beta$, where $s \in S_n$ is an optimal reply for player $n$ against $\sigma$. A profile $\tau$ is a $\beta$-reply against $\sigma$ if for each $n$ the strategy $\tau_n$ is a $\beta$-reply for player $n$ against $\sigma$.

**Step 1.** First we show that without loss of generality we can assume that $G$ satisfies the following property (*): for every neighborhood $W$ of Graph(BR) there exists a map $h : \Sigma \to \Sigma$ such that:

1. Graph$(h) \subset W$.
2. For each player $n$ the $n$-th coordinate map $h_n$ of $h$ depends only on $\Sigma_{-n}$.
3. $h$ has no fixed points in $U$.

It suffices to show that there exists an equivalent game $G^*$ that satisfies (*).

Define $G^*$ as follows. Player $n$'s pure strategy set is $S_n^* = S_n \times S_{n+1}$, where $n+1$ is taken modulo $N$. For each $n$ and $m \in \{n, n+1\}$ denote by $p_{n,m}$ the natural projection from $S_n^*$ to $S_m$. Then the payoff function for player $n$ is given by $G_n^*(s^*) = G_n(s)$, where for each $m$, $s_m = p_{m,m}(s_m^*)$. In other words, $n$'s choice of a strategy for $n+1$ is payoff irrelevant. Clearly $G^*$ is equivalent to $G$. Let $\Sigma_n^*$ be player $n$'s set of mixed strategies in the game $G^*$. We continue to use $p_{n,m}$ to denote the function from $\Sigma_n^*$ to $\Sigma_m$ that computes for each mixed strategy $\sigma_n^*$ the induced marginal distribution over $S_m$. Let $p : \Sigma^* \to \Sigma$ be the function $p(\sigma^*) = (p_{1,1}(\sigma_1^*), \ldots p_{N,N}(\sigma_N^*))$; i.e., $p$ computes the payoff-relevant coordinates of $\sigma^*$. Finally let $P : \Sigma^* \times \Sigma^* \to \Sigma \times \Sigma$ be the function for which $P(\sigma^*, \tau^*) = (p(\sigma^*), p(\tau^*))$. Use BR$^*$ to denote the best-reply correspondence for the game $G^*$. Similarly $C^*$ denotes the component of equilibria of $G^*$ that are equivalent to equilibria in $C$, and $U^*$ denotes the neighborhood corresponding to $U$.

Fix a neighborhood $W^*$ of Graph(BR$^*$). For each $\mu > 0$, let $W(\mu)$ be the set of those $(\sigma, \tau) \in \Sigma \times \Sigma$ for which $\tau$ is a $\mu$-reply to $\sigma$ in $G$. Then the collection $\{W(\mu) \mid \mu > 0\}$ is a basis of neighborhoods of the graph of BR. Choose $\mu > 0$ such that $P^{-1}(W(\mu)) \subseteq W^*$. Corollary A.2 in the Appendix shows that, since $C$ has index zero, there exists a function $h : \Sigma \to \Sigma$ such that the graph of $h$ is contained in $W(\mu)$ and $h$ has no fixed points in $U$. Now define the map $h^* : \Sigma^* \to \Sigma^*$ as follows: for each $n$, $h_n^*(\sigma^*)$ is the product distribution $\tau_n(\sigma^*) \times p_{n+1,n+1}(\sigma_{n+1}^*)$, where

$$\tau_n(\sigma^*) = h_n(p_{1,1}(\sigma_1^*), \ldots p_{n-1,n-1}(\sigma_{n-1}^*), p_{n-1,n}(\sigma_{n-1}^*), p_{n+1,n+1}(\sigma_{n+1}^*), \ldots, p_{N,N}(\sigma_N^*)).$$

By construction, each coordinate map $h_n^*$ depends only on $\Sigma_{-n}^*$. We claim that the graph of $h^*$ is contained in $W^*$. To see this remark first that $\tau_n(\sigma^*)$ is player $n$'s component of the image of $(p_{-n}(\sigma^*), p_{n-1,n}(\sigma_{n-1}^*))$ under $h$. Since the graph of $h$ is a subset of $W(\mu)$, $\tau_n^*$ is a $\mu$-reply to $p_{-n}(\sigma^*)$. Therefore, $(p(\sigma^*), \tau(\sigma^*))$ belongs to $W(\mu)$. Hence $(\sigma^*, h^*(\sigma^*)) \in P^{-1}(W(\mu)) \subseteq W^*$.

To finish the proof we show that $h^*$ has no fixed point in $U^*$. Suppose $\sigma^*$ is a fixed point of $h^*$. Then each $\sigma^*_n$ is a product distribution with $p_{n,n+1}(\sigma^*_n) = p_{n+1,n+1}(\sigma^*_{n+1})$ for all $n$. Therefore

$$p_{n,n}(\sigma^*_n) = p_{n,n}(h^*_n(\sigma^*)) = h_n(p_{-n}(\sigma^*), p_{n-1,n}(\sigma^*_{n-1}) = h_n(p(\sigma^*))$$

for each player $n$, which implies that $p(\sigma^*)$ is a fixed point of $h$. Since $h$ has no fixed point in $U$, $\sigma^* \notin U^*$.

**Step 2.** Let $I$ be the interval $[-\delta/2, \delta/2]$. We now show that without loss of generality we can assume that $G$ satisfies the following property (**): there exists a function $g : \Sigma \to I^R$, where $R = \sum_n |S_n|$, such that:

1. For each player $n$, $g_n$ depends only on $\Sigma_{-n}$.
2. No profile $\sigma \in U$ is an equilibrium of the game $G \oplus g(\sigma)$.

As in Step 1 we prove this by constructing an equivalent game with the property (**). Since the payoff functions are multilinear on the compact set $\Sigma$, there exists a (Lipschitz) constant $M > 0$ such that $\|G_n(\sigma) - G_n(\tau)\| \leqslant M\|\sigma - \tau\|$ for all $n$ and $\sigma, \tau \in \Sigma$. We begin with a preliminary lemma.

**Lemma 4.3.** *If $\tau_n$ is a $\beta_1$-reply to $\sigma$ and $\|\sigma' - \sigma\| \leqslant \beta_2$ then $\tau_n$ is a $(2M\beta_2 + \beta_1)$-reply to $\sigma'$.*

*Proof of the Lemma.* Let $s$ be an optimal reply for player $n$ to $\sigma'$. Then the result follows from the following inequality:

$$G_n(s, \sigma'_{-n}) - G_n(\tau_n, \sigma'_{-n}) \leqslant |G_n(s, \sigma'_{-n}) - G_n(s, \sigma_{-n})| + G_n(s, \sigma_{-n}) - G_n(\tau_n, \sigma_{-n}) + |G_n(\tau_n, \sigma_{-n}) - G_n(\tau_n, \sigma'_{-n})|.$$

□

Fix $\eta = \delta/16M$. For each $\sigma \in S$ there exists an open ball $B(\sigma)$ around $\sigma$ of radius less than $\eta$ such that for each $\sigma' \in B(\sigma)$ the set of pure best replies against $\sigma'$ is a subset of those that are best replies to $\sigma$. Since the set of best replies for each player $n$ to a strategy profile is the face of $\Sigma_n$ spanned by his pure best replies, $\mathrm{BR}(\sigma') \subseteq \mathrm{BR}(\sigma)$ for each $\sigma' \in B(\sigma)$. The balls $B(\sigma)$ define an open covering of $\Sigma$. Hence there exists a finite set of points $\sigma^1, \ldots, \sigma^k$ whose corresponding balls form a subcover. For each $\sigma^i$ choose an $\eta$-neighborhood $W(\sigma^i)$ of $\mathrm{BR}(\sigma^i)$. Let $W = \cup_i B(\sigma^i) \times W(\sigma^i)$. Then $W$ is a neighborhood of the graph of BR. From Step 1 there exists a function $h : \Sigma \to \Sigma$ such that (1) Graph$(h) \subset W$; (2) for each $n$, $h_n$ depends only on $\Sigma_{-n}$; and (3) $h$ has no fixed point in $U$. If $\tau = h(\sigma)$ then there exist $\sigma^i$, $\tau^i$ such that $\sigma \in B(\sigma^i)$, $\tau^i$ is a best reply to $\sigma^i$, and $\tau$ is within $\eta$ of $\tau^i$. Therefore, the Lemma implies that $\tau^i$ is a $2M\eta$-reply against $\sigma$; and using the Lipschitz inequality, $\tau$ is a $3M\eta$-reply against $\sigma$.

Fix $\alpha > 0$ such that if $\sigma \in U$ then $\|\sigma - h(\sigma)\| > \alpha$. Take a sufficiently fine subdivision of each strategy simplex $\Sigma_n$ such that the diameter of each simplex is less than both $\eta$ and $\alpha$. Call this simplicial complex $\mathcal{T}_n$; then $|\mathcal{T}_n|$ is the simplex $\Sigma_n$ viewed as the space of the simplicial complex $\mathcal{T}_n$. Let $\mathcal{T}$ be the multisimplicial complex $\prod_n \mathcal{T}_n$ composed of products of simplices, and let $\mathcal{T}_{-n} = \prod_{m \neq n} \mathcal{T}_m$. By Corollary B.5, for each player $n$ and each sufficiently fine subdivision of the multisimplicial complex $\mathcal{T}_{-n}$, there exists a multisimplicial approximation to $h_n$. Therefore, there exists a subdivision of each $\mathcal{T}_m$, say $\mathcal{T}^*_m$, such that for each player $n$ the map $h_n : |\mathcal{T}_{-n}| \to |\mathcal{T}_n|$ has a multisimplicial approximation $h^*_n : |\mathcal{T}^*_{-n}| \to |\mathcal{T}_n|$. Denote by $h^*$ the induced multisimplicial map.

Let $T_n$ and $T^*_n$ be the sets of vertices of $\mathcal{T}_n$ and $\mathcal{T}^*_n$, respectively, and define $T = \prod_n T_n$. We now define a game $\overline{G}$ that is equivalent to $G$, as follows. For each player $n$ the set of pure strategies is $T_n$. The pure strategy $t_n \in T_n$ is a duplicate of the mixed strategy in $\Sigma_n$ corresponding to the vertex $t_n$ of $\mathcal{T}_n$. Since the vertices of $\Sigma_n$ belong to $T_n$, $\overline{G}$ is equivalent to $G$. Let $\overline{\Sigma}_n$ be the set of mixed strategies of player $n$ in $\overline{G}$ and let $\overline{\Sigma} = \prod_n \overline{\Sigma}_n$.

We now construct a function $g : \overline{\Sigma} \to \prod_n \mathbb{R}^{T_n}$ with the requisite properties by first defining $g$ on $\Sigma$ and then extending it to the whole of $\overline{\Sigma}$ by letting $g(\overline{\sigma})$ be $g(\sigma)$, where $\sigma$ is the equivalent profile in $G$. For each $n$ let $f_n : \Sigma_{-n} \to \mathbb{R}$ be the function defined by $f(\sigma_{-n}) = \max_{s \in S_n} G_n(s, \sigma_{-n})$. For each $t_n \in T_n$ let $\mathcal{V}(t_n)$ be the set of vertices $v_{-n}$ of $\mathcal{T}^*_{-n}$ such that $h^*_n(v_{-n}) = t_n$. Also, let $X(t_n)$ the set of $\sigma_{-n}$ whose distance from $X(t_n)$ is at most $\eta$, and let $Y(t_n)$ be the set of points whose distance from $\mathcal{V}(t_n)$ is at least $2\eta$. If $\mathcal{V}(t_n)$ is nonempty, using Urysohn's Lemma, define a function $\pi_{t_n} : |\mathcal{T}^*_{-n}| \to [0, 1]$ whose value is 1 on $X(t_n)$ and 0 on $Y(t_n)$. Otherwise, define $\pi_{t_n}$ to the zero function. Since the diameter of each multisimplex is at most $\eta$, for each vertex $v_{-n} \in \mathcal{V}(t_n)$, $\pi_{t_n}(\cdot) = 1$ on $\mathrm{Cl}\,\mathrm{St}(v_{-n})$. For each $n$ define $g^1_{t_n} : \Sigma_{-n} \to \mathbb{R}^{T_n}$ by letting $g^1_{t_n}(\sigma_{-n}) = \pi_{t_n}(\sigma_{-n})(f_n(\sigma_{-n}) - G_n(t_n, \sigma_{-n}))$. Define $g^2_{t_n} : \Sigma_{-n} \to \mathbb{R}^{T_n}$ as follows. If $\mathcal{V}(t_n)$ is empty, then it is the zero function. Otherwise for each $\sigma_{-n}$,

$$g^2_{t_n}(\sigma_{-n}) = (2M\eta/|\mathcal{V}(t_n)|) \sum_{v_{-n} \in \mathcal{V}(t_n)} \prod_{m \neq n} \sigma_m(v_m)\,,$$

where for each $\sigma_m$ and each vertex $v_m$ of $\mathcal{T}^*_m$, $\sigma_m(v_m)$ is the $v_m$-th barycentric coordinate of $\sigma_m$. Finally, let $g_{t_n}(\sigma_{-n}) = g^1_{t_n}(\sigma_{-n}) + g^2_{t_n}(\sigma_{-n})$. For each $t_n$ and each $v_{-n}$ that is mapped to $t_n$ by $h^*_n$, we have that $t_n$ is within $\eta$ of $h_n(v_{-n})$. Since the latter is a $3M\eta$-reply against $v_{-n}$, $t_n$ is a $4M\eta$-reply against $v_{-n}$. Hence $t_n$ is a $6M\eta$-reply against every $\sigma_{-n} \in \Sigma_{-n} \backslash Y(t_n)$. Finally, $g^2_{t_n}(\cdot) \leqslant 2M\eta$ implies that $\|g_{t_n}(\sigma_{-n})\| \leqslant 8M\eta = 8M\delta/16M = \delta/2$ for each $\sigma_{-n} \in \Sigma_{-n}$. Thus $g_n$ maps $\Sigma_{-n}$ into $I^{T_n}$. Obviously the extension of $g$ to the whole of $\overline{\Sigma}$ also has norm at most $\delta/2$.

To finish the proof of this step we show that if $\overline{\sigma} \in \overline{U}$ then $\overline{\sigma}$ is not an equilibrium of $\overline{G} \oplus g(\overline{\sigma})$. Suppose to the contrary that $\overline{\sigma} \in \overline{U}$ is such an equilibrium and let $\sigma$ be the corresponding strategy in $\Sigma$. For each $n$ let $K^*_n$ be the simplex of $\mathcal{T}^*_n$ that contains $\sigma_n$ in its interior. Let $L_n$ be the simplex of $\mathcal{T}_n$ spanned by the images of the vertices of $K^*_{-n}$ under the map $h^*_n$. Observe that for each $t_n$ that is a vertex of $L_n$, $G_n(t_n, \sigma_{-n}) + g^1_{t_n}(\sigma_{-n}) = f_n(\sigma_{-n})$ and $g^2_{t_n}(\sigma_{-n}) > 0$, while for any other vertex $t_n$ in $T_n$, $G_n(t_n, \sigma_{-n}) + g^1_{t_n}(\sigma_{-n}) \leqslant f_n(\sigma_{-n})$ and $g^2_{t_n}(\sigma_{-n}) = 0$. Hence, the pure best replies against $\sigma_{-n}$ (and therefore against $\overline{\sigma}_{-n}$) are a subset of $L_n$. To derive a contradiction, it suffices to show that $\sigma \notin L$, where $L$ is the multisimplex $\prod_n L_n$. Let $L'_n$ be the simplex of $\mathcal{T}_n$ containing $h_n(\sigma)$ in its interior, and let $L' = \prod_n L'_n$. Since for each $n$, $h^*_n$ is a multisimplicial approximation of $h_n$, $L_n$ is a face of $L'_n$ and thus $L$ is a face of the multisimplex $L'$. Since the diameter of each multisimplex of $\mathcal{T}$ is less than $\alpha$ and since $\|\sigma - h(\sigma)\| > \alpha$, $\sigma \notin L'$ and *a fortiori* $\sigma \notin L$. This concludes the proof of Step 2.

**Step 3.** Suppose $g : \Sigma \to I^R$ has the property (**) described in Step 2. For each $\sigma \in U$ there exists $\zeta(\sigma) > 0$ and an open ball $B(\sigma)$ around $\sigma$ such that for each $\sigma' \in B(\sigma)$ and each $g'$ such that $\|g' - g(\sigma')\| \leqslant \zeta(\sigma)$, $\sigma'$ is not an equilibrium of $G \oplus g'$. The balls $B(\sigma)$ form an open covering of $U$. Hence there exists a finite set of points $\sigma^1, \ldots, \sigma^k$ such that their corresponding balls cover $\Sigma$. Let $\zeta = \min_i \zeta(\sigma^i)$. Construct a subdivision $\mathcal{I}$ of the interval $I$ such that the diameter of each simplex (i.e., a subinterval) is at most $\zeta$. Once again, using the multisimplicial approximation theorem, Theorem B.4 in the Appendix, there exists a simplicial subdivision $\mathcal{T}_n$ of each $\Sigma_n$ and for each $s \in S_n$, a multisimplicial approximation $g^*_s : |\mathcal{T}_{-n}| \to |\mathcal{I}|$ of $g_s$ that is multilinear on each multisimplex of $\mathcal{T}_{-n}$. Let $g^* : \Sigma \to |\mathcal{I}|^R$ be the corresponding multisimplicial function defined by the coordinate functions $g^*_s$. By construction, no $\sigma \in U$ is an equilibrium of $G \oplus g^*(\sigma)$.

As in Section B.2 of the Appendix, let $\mathcal{P}_n$ be the polyhedral complex generated by $\mathcal{T}_n$, and let $\gamma_n : \Sigma_n \to \mathbb{R}$ be the associated convex function. For each $n$ let $P_n$ be the set of vertices of $\mathcal{P}_n$. Given a polyhedron $P_{-n}$ in $\prod_{m \neq n} \mathcal{P}_m$, there exists a multisimplex $T_{-n}$ of $\mathcal{T}_{-n}$ that contains it. Since $g^*_s$ is multilinear on each multisimplex, $g^*$ is multilinear on each polyhedron.

Consider now the equivalent game $\tilde{G}$ where the strategy set of each player $n$ is the set $P_n$ of vertices of the polyhedral complex $\mathcal{P}_n$. Let $\tilde{\Sigma}_n$ be the set of mixed strategies of player $n$ in the game $\tilde{G}$. For each player $n$, let $A_n$ be the $|S_n| \times |P_n|$ matrix, where column $p$ is the mixed strategy vector that corresponds to the vertex $p$ of $\mathcal{P}_n$. Then the payoff to player $n$ from a strategy vector $\tilde{\sigma} \in \tilde{\Sigma}$ is his payoff in $G$ from the profile $\sigma$, where $\sigma_m = A_m \tilde{\sigma}_m$ for each $m$. For each $n$, let $B_n : P_{-n} \to I^{P_n}$ be the function defined by $B_n(p_{-n}) = A_n g^*_n(p_{-n})$. Consider now the game $\tilde{G}'$ obtained by modifying the payoff functions to the following: for each player $n$, his payoff from the pure-strategy profile $p$ is $\tilde{G}_n(p) + B_{n,p_n}(p_{-n})$. By construction $\tilde{G}'$ is a $\delta/2$-perturbation of $\tilde{G}$. Let $c_n$ be the vector in $\mathbb{R}^{P_n}$ where the coordinate $p$ of $c_n$ is $\gamma_n(p)$. For each $\delta' \leqslant \delta/2$ let $\tilde{G}_{\delta'}$ be the game $\tilde{G}' \oplus [-\delta' c]$. Then $\tilde{G}_{\delta'}$ is a $\delta$-perturbation of $\tilde{G}$.

We claim now that for sufficiently small $\delta'$, the game $\tilde{G}_{\delta'}$ has no equilibrium in the set $\tilde{U}$ that is the corresponding neighborhood of the component $\tilde{C}$ in the game $\tilde{G}$ that is equivalent to $C$. Indeed, suppose to the contrary that there is a sequence $\delta^k$ converging to zero and a corresponding sequence $\tilde{\sigma}^k$ of equilibria of $\tilde{G}_{\delta^k}$ that lie in $\tilde{U}$. For each $k$ let $\sigma^k$ be the equivalent profile in $\Sigma$. For each $k$ and each player $n$, if $\tilde{\tau}^k_n$ is a mixed strategy such that $A_n \tilde{\tau}^k_n = \sigma^k_n$ then $c'_n \tilde{\tau}^k_n \geqslant c'_n \tilde{\sigma}^k_n$. Thus $\tilde{\sigma}^k_n$ solves the linear programming problem $\min c'_n \tilde{\tau}^k_n$ subject to $A_n \tilde{\tau}^k_n = \sigma^k_n$. Let $L^k_n$ be the unique polyhedron of $\mathcal{P}_n$ that contains $\sigma^k_n$ in its interior. Since $\gamma_n$ is a convex function, $\gamma_n(\sigma^k) \leqslant \sum_{p_n \in P_n} \gamma_n(A_{n,p_n})' \tilde{\tau}^k_{n,p_n}$ for all $\tilde{\tau}^k_n$ such that $A_n \tilde{\tau}^k_n = \sigma^k$, where $A_{n,p_n}$ is the $p$-th column of $A_n$ and $\tilde{\tau}^k_{n,p_n}$ is the probability that $\tilde{\tau}^k_n$ assigns to the pure strategy $p_n$. Moreover, since $\gamma_n$ is linear exactly on the polyhedra of $\mathcal{P}_n$, this inequality is strict unless the support of $\tilde{\tau}^k_n$ is contained in the set of vertices of $L^k_n$. Therefore, the equilibrium $\tilde{\sigma}^k$ assigns positive probability only to the vertices of $L^k_n$.

Now let $\tilde{\sigma}$ be a limit of $\tilde{\sigma}^k$ as $\delta^k \downarrow 0$ and let $\sigma$ be the equivalent mixed strategy. Then $\tilde{\sigma}$ is an equilibrium of the game $\tilde{G}'$. Therefore, $\sigma$ is an equilibrium of the game $G \oplus b$, where $b_{n,s} = \sum_{p_{-n} \in P_{-n}} g^*_{n,s}(p) \prod_{m \neq n} \tilde{\sigma}_{m,p}$ for each $n$ and $s \in S_n$. By the arguments in the previous paragraph, there exists for each $n$ a polyhedron $P^\circ_n \in \mathcal{P}_n$ such that $\tilde{\sigma}_n$ assigns positive probability only to a subset of the vertices of $P^\circ_n$. Since each $g^*_n$ is multilinear on the multisimplex $T_{-n}$ that contains $P^\circ_{-n}$, $b_{n,s} = g^*_{n,s}(\sigma_{-n})$. Thus $\sigma$ is an equilibrium of $G \oplus g(\sigma)$, which is a contradiction. Thus, for all sufficiently small $\delta'$ the game $\tilde{G}_{\delta'}$ has no equilibrium in $\tilde{U}$. □

## 5. Concluding Remarks

Theorem 1.1 has implications for the stronger definition of stability proposed by Mertens (1989). Its key additional requirement is that the local degree of the projection map $p : \mathcal{E} \to \mathcal{G}$ from the equilibrium graph to the space of games should be nonzero; that is, the projection map is locally essential or "nontrivial" in the terminology of algebraic topology. Govindan and Wilson (1997) show that the index of a component is the same as the degree of this projection map from any sufficiently small neighborhood of the component. Theorem 1.1 therefore implies that a hyperstable component satisfies this requirement.

**Corollary 5.1.** *An equilibrium component is hyperstable if and only if the projection map $p : \mathcal{E} \to \mathcal{G}$ is locally essential.*

However, Mertens' definition implies stronger properties. For instance, Govindan and Wilson (2001b) prove that a maximal Mertens' stable set is a component of the perfect equilibria, as defined by Selten (1975), whereas KM observe that a hyperstable component can contain equilibria that use weakly dominated strategies, and in particular, equilibria that are not perfect.

Theorem 1.1 is the analog for games of a classic theorem of algebraic topology derived from the Hopf Extension Theorem (Spanier, 1966, 8.1.18). Say that a component of the fixed points of a map from a space into itself is essential if every nearby map has a nearby fixed point. For a class of spaces that includes manifolds with boundaries, O'Neill (1953) proves that a component is essential if and only if its index is nonzero; McLennan (1988, Appendix E) provides an alternative proof. However, Dold (1972, VII.6.25.4) describes a space where O'Neill's construction fails.

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## Appendix A. Index Theory

A.1. **An Index Derived from the Best-Reply Correspondence.** In this section we define an index for components of equilibria using the best-reply correspondence. Theorem A.1 shows that this index coincides with the standard index (e.g., Gül, Pearce, and Stacchetti, 1993; Govindan and Wilson, 1997) constructed from a map. The notation conforms to Section 2.

Let $\mathrm{BR} : \Sigma \Rightarrow \Sigma$ be the best-reply correspondence for the game $G$, i.e., $\mathrm{BR}(\sigma) = \{\tau \in \Sigma \mid (\forall n)\ \tau_n \in \arg\max_{\tilde{\tau}_n} \tilde{\tau}_n' \cdot G_n(\sigma_{-n})\}$. The set $E$ of equilibria of $G$ is the set of fixed points of BR; i.e., those for which $\sigma \in \mathrm{BR}(\sigma)$. Let $C$ be a component of the equilibria of $G$. We follow McLennan (1989) in defining an index for $C$. Let $U$ be an open neighborhood of $C$ such that its closure $\overline{U}$ satisfies $\overline{U} \cap E = C$. Let $W$ be a neighborhood of Graph(BR) such that $W \cap \{(\sigma,\sigma) \in \Sigma \times \Sigma \mid \sigma \in \overline{U} - U\} = \emptyset$. By Corollary 2 in McLennan (1989) there exists a neighborhood $V \subseteq W$ of Graph(BR) such that if $f_0$ and $f_1$ are any two maps from $\Sigma$ to $\Sigma$ whose graphs are contained in $V$, then there is a homotopy $F : [0,1] \times \Sigma \to \Sigma$ from $f_0$ to $f_1$ such that $\mathrm{Graph}(F) \subset [0,1] \times V$. By the Proposition in McLennan (1989) there exists a map $f : \Sigma \to \Sigma$ for which $\mathrm{Graph}(f) \subset V$. Define the index $\mathrm{Ind}_{\mathrm{BR}}(C)$ to be the standard index of the restricted map $f : U \to \Sigma$; i.e., $\mathrm{Ind}_{\mathrm{BR}}(C)$ is the degree of the corresponding displacement map $\mathrm{Id} - f$. The choice of the neighborhood $V$ ensures that this index does not depend on the particular map $f$ chosen to compute the index (Dold, 1972).

The index of the component $C$ can also be defined using the index obtained from the map $g : \Sigma \to \Sigma$ defined by Gül, Pearce, and Stacchetti (1993), as follows. Let $Z = \prod_n \mathbb{R}^{S_n}$ and define the map $w : \Sigma \to \mathcal{Z}$ by $w(\sigma) = z$, where $z_{n,s} = \sigma_n(s) + G_n(s, \sigma_{-n})$ for each player $n$ and each pure strategy $s \in S_n$. Also, let $r : \mathcal{Z} \to \Sigma$ be the retraction that maps a point $z$ to the point $\tau \in \Sigma$ that is closest to $z$ in $\ell_2$-distance. Specifically, $r(z)$ is computed as follows: for each player $n$, define $v_n(z)$ to be the unique scalar $\alpha$ such that $\sum_{s \in S_n} (z_{n,s} - \alpha)^+ = 1$; then $r(z)_{n,s} = (z_{n,s} - v_n(z))^+$ for each $n$ and $s \in S_n$. Finally $g = r \circ w$. The equilibria of $G$ are precisely the fixed points of $g$. Define the Gül-Pearce-Stacchetti index $\mathrm{Ind}_{\mathrm{GPS}}(C)$ to be the standard index of the component $C$ computed from the map $g : U \to \Sigma$.

**Theorem A.1.** *$\mathrm{Ind}_{BR}(C) = \mathrm{Ind}_{GPS}(C)$.*

*Proof.* For each $\lambda > 0$ define the game $G^\lambda$ as the game where the payoff functions of all players in $G$ are multiplied by $\lambda$; i.e., $G^\lambda = \lambda G$. Clearly, all games $G^\lambda$ have the same equilibria. For $G^\lambda$ let $w^\lambda$ be the map corresponding to $w$ in the game $G$, and let $g^\lambda = r \circ w^\lambda$ be the corresponding GPS map. Then for each $\lambda > 0$ the homotopy $H : [0,1] \times \Sigma \to \Sigma$, $H(t,\sigma) = g^{1+t(\lambda-1)}$, from $g$ to $g^\lambda$ preserves the set of fixed points. Hence, the index of $C$ under $g^\lambda$ is the same for all $\lambda$. To prove Theorem A.1 it is sufficient to show that there exists $\lambda > 0$ such that the graph of $g^\lambda$ is contained in $V$.[2] For each $\lambda > 0$ and $\sigma \in \Sigma$, $w^\lambda(\sigma) \equiv z^\lambda$ is such that $1 + \lambda G_n(s, \sigma_{-n}) \geqslant z^\lambda_{n,s} \geqslant \lambda G_n(s, \sigma_{-n})$ for all $n, s$. Choose $c(\sigma) > 0$ such that if $s$ is not a best reply to $\sigma_{-n}$ for player $n$, then $G_n(s', \sigma_{-n}) - G_n(s, \sigma_{-n}) \geqslant c(\sigma)$, where $s'$ is a best reply for player $n$ against $\sigma$. Then $\sum_{s' \in \mathrm{BR}_n(\sigma)} z^\lambda_{n,s'} - z^\lambda_{n,s} \geqslant \lambda c(\sigma) - 1$ if $s$ is not a best reply. In particular, if $\lambda \geqslant 2/c(\sigma)$, then this difference is at least 1. Therefore, for each such $\lambda$, $z^\lambda$ is retracted by $r$ to a point in $\mathrm{BR}(\sigma)$. Now choose an open ball $B(\sigma)$ around $\sigma$ in $\Sigma$ such that (i) $B(\sigma) \times \mathrm{BR}(\sigma) \subset V$; and (ii) $G_n(s', \sigma'_{-n}) - G_n(s, \sigma'_{-n}) \geqslant c(\sigma)/2$ for each $n$, $s \notin \mathrm{BR}(\sigma)$, and $s' \in \mathrm{BR}(\sigma)$. Then as before, $g^\lambda(\sigma) \in \mathrm{BR}(\sigma)$ for each $\lambda \geqslant 4/c(\sigma)$ and $\sigma' \in B(\sigma)$. The balls $B(\sigma)$ for $\sigma \in \Sigma$ form an open cover of $\Sigma$. Since $\Sigma$ is compact there exists a finite set $\sigma^1, \ldots, \sigma^K \in \Sigma$ such that $\cup_k B(\sigma^k) \supset \Sigma$. Let $\lambda^* = \max_k 4/c(\sigma^k)$. For each $\lambda \geqslant \lambda^*$ the graph of $g^\lambda$ belongs to $V$. $\square$

A corollary follows from McLennan (1988, 4.4, Theorem 6).

[2]We actually prove a stronger statement: $\mathrm{Graph}(g^\lambda) \subset V$ for sufficiently large $\lambda$. In fact, the proof shows that each neighborhood of Graph(BR) contains $\mathrm{Graph}(g^\lambda)$ for all sufficiently large $\lambda$. Thus as $\lambda \uparrow \infty$, $\mathrm{Graph}(g^\lambda)$ converges to Graph(BR) in the upper topology, as defined by McLennan (1988). Put differently, if $\lambda$ is large then $g^\lambda$ is a good approximation of the best-reply correspondence.

**Corollary A.2.** *If $\mathrm{Ind}_{BR}(C) = 0$ then for each neighborhood $V$ of Graph$(BR)$ there exists a map $h : \Sigma \to \Sigma$ such that Graph$(h) \subset V$ and $h$ has no fixed point in the neighborhood $U$ of $C$.*

A.2. **Equivalence of Index and Degree.** Let $\Gamma = \mathbb{R}^{N|S|}$ be the space of all finite $N$-player games with a fixed strategy set $S_n$ for each player, and $S = \prod_n S_n$. Let $\mathcal{E}^*$ be the graph of the Nash equilibrium correspondence over $\Gamma$. Each game can be written uniquely as a pair $(\tilde{G}, g)$ where for each player $n$ and each pure strategy $s \in S_n$, $\sum_{s_{-n}} G_s(s_{-n}) = 0$. Thus, $\Gamma$ is the product space $\tilde{Z} \times \mathcal{Z}$ of all pairs $(\tilde{G}, g)$. KM show that there exists a homeomorphism $\Theta : \mathcal{E}^* \to \tilde{Z} \times \mathcal{Z}$ such that $p \circ \Theta^{-1}$ is homotopic to a homeomorphism that extends to the one-point compactification of $\mathcal{G}$. In particular the map $p \circ \Theta^{-1}$ has degree $+1$. We can therefore orient $\mathcal{E}^*$ such that the projection map has degree 1. Given a game $G$ and a component $C$ of the game, choose a neighborhood $U$ of $\{\tilde{G}, g)\} \times C$ in the graph that is disjoint from the other components of the set of equilibria of $G$ (viewed as a subset of $\mathcal{E}^*$). The degree of $C$, denoted deg$(C)$ is the local degree of $p$ over $U$. Since $\Theta$ is the identity on the $\tilde{\mathcal{Z}}$ factor, we can also define the degree of $C$ using $Z$ as the space of games. Indeed given a game $G = (\tilde{G}, g)$, let $\mathcal{E} = (g', \sigma)$ such that $((\tilde{G}, g'), \sigma)$ belongs to $\mathcal{E}^*$. Let $\theta : \mathcal{E} \to \mathcal{Z}$ be the map $\theta(g', \sigma) = z$, where $z$ is such that $\Theta((\tilde{G}, g'), \sigma) = (\tilde{G}, z)$. Then $\theta$ is a homeomorphism between $\mathcal{E}$ and $\mathcal{Z}$ and as before we can define the degree of $C$ as the local degree of the projection map from a neighborhood $U$ of $\{g\} \times C$ in $\mathcal{E}$. Obviously, these two definitions are equivalent. If we use $\theta$, then degree of $C$ is just the degree of $g$ under the map $f' \equiv p \circ \theta^{-1}$ from $V = \theta(U)$ in $\mathcal{Z}$, where $p$ is the natural projection from $\mathcal{E}$ to $\mathcal{Z}$. Let $f : \mathcal{Z} \to \mathcal{Z}$ be the map $f(z) = f'(z) - g$. Then the degree of zero under the map $f$ over $U$ is the same as the degree of the map $f'$ over $U$. As we saw in Section 2, the degree of $f$ over $U$ is the index of $F$ over $U$, which is the same as the index of the GPS map.

A.3. **Invariance of Index and Degree.** Govindan and Wilson (1997) show that the index (and hence also the degree) of a component of equilibria is invariant under the addition or deletion of duplicate strategies. The proof there is incomplete to the extent that it holds only when duplicates of pure strategies are added. While the proof can be extended to more general duplicates, we present here a simple proof using the index defined using the best-reply correspondence.

**Theorem A.3.** *The index of a component of equilibria is invariant under the addition of duplicate strategies.*

*Proof.* Let $C$ be a component of equilibria of a game $G$. It suffices to show that the index of $C$ is invariant under the addition of duplicate strategies. Accordingly, for each player $n$ let $T_n$ be a finite collection of mixed strategies. Let $G^*$ be the game obtained by adding the strategies in $T_n$ as pure strategies for $n$; i.e., $n$'s pure strategy set in $G^*$ is $S_n \cup T_n$. Let $\Sigma_n^*$ be his set of mixed strategies. Let BR$^*$ be the best-reply correspondence in $\Sigma^*$. Let $p^* : \Sigma^* \to \Sigma$ be the function that maps each mixed strategy in $G^*$ to the equivalent mixed strategy in $G$. Let $\iota : \Sigma \to \Sigma^*$ be the "inclusion" map that sends a point in $\Sigma$ to the corresponding point on the face of $\Sigma^*$. More precisely, $\iota(\sigma) = \sigma^*$, where $\sigma^*_{n,s} = \sigma_{n,s}$ for $s \in S_n$ and $\sigma_{n,t} = 0$ for $t \in T_n$. Obviously, $\iota(\sigma) \subset p^{-1}(\sigma)$ for each $\sigma \in \Sigma$.

Let $C^* \equiv p^{-1}(C)$ be the component of equilibria of $G^*$ corresponding to $C$. Let $U$ be an open neighborhood of $C$ whose closure is disjoint from the other components of equilibria of $G$. Let $U^* = p^{-1}(U)$. Choose a neighborhood $W^*$ of the graph of BR$^*$ such that the index of $C^*$ can be computed as the sum of the indices of the fixed points in $U^*$ of any function $h^*$ whose graph is contained in $W^*$.

Let $W$ be a neighborhood of the graph of BR such that $(\sigma, \tau) \in W$ implies $p^{-1}(\sigma) \times p^{-1}(\tau) \subset W^*$. By the definition of $\mathrm{Ind}_{\mathrm{BR}}(C)$, there exists a function $h : \Sigma \to \Sigma$ such that (i) the graph of $h$ is contained in $W$; (ii) $h$ has no fixed points on the boundary of $U$; and (iii) $\mathrm{Ind}_{\mathrm{BR}}(C)$ is the index of the map $h$ over $U$. Define now a map $h^* : \Sigma^* \to \Sigma^*$ by $h^* = \iota \circ h \circ p$. Then, by construction the graph of $h^*$ is contained in $W^*$.

Moreover, $h$ and $h^*$ have homeomorphic sets of fixed points. In fact, the fixed points of $h^*$ are the image of the fixed points of $h$ under the injective map $\iota$. Moreover, letting $h^0 = \iota \circ h$, we have that $h = p \circ h^0$ and $h^* = h^0 \circ p$. Therefore, by the commutativity property of index (cf., Dold, 1972, VII.5.9), the index of each component $F$ of the set of fixed points of $h$ is the same as the index of $\iota(F)$. Hence $\mathrm{Ind}_{\mathrm{BR}^*}(C^*) = \mathrm{Ind}_{\mathrm{BR}}(C)$. □

## Appendix B. Multisimplicial Approximation

B.1. **A Multisimplicial Approximation Theorem.** The purpose of this section is to establish a multilinear version of the Simplicial Approximation Theorem. Perhaps this result is well known but we have not found a reference in the literature. We begin with some definitions; cf. Spanier (1966, Chapter 3) for details.

A set of points $\{v_0, \ldots, v_n\}$ in $\mathbb{R}^N$ is affinely independent if the equations $\sum_{i=0}^{n} \lambda_i v_i = 0$ and $\sum_i \lambda_i = 0$ imply that $\lambda_0 = \cdots = \lambda_n = 0$. An $n$-simplex $K$ in $\mathbb{R}^N$ is the convex hull of an affinely independent set $\{v_0, \ldots, v_n\}$. Each $v_i$ is a vertex of $K$ and the collection of vertices is called the vertex set of $K$. Each $\sigma \in K$ is expressible as a unique convex combination $\sum_i \lambda_i v_i$; and for each $i$, $\sigma(v_i) \equiv \lambda_i$ is the $v_i$-th barycentric coordinate of $\sigma$. The interior of $K$ is the set of $\sigma$ such that $\sigma(v_i) > 0$ for all $i$. A face of $K$ is the convex hull of a subset of the vertex set of $K$.

A (finite) simplicial complex $\mathcal{K}$ is a finite collection of simplices such that the face of each simplex in $\mathcal{K}$ belongs to $\mathcal{K}$, and the intersection of two simplices is a face of each of the two. The set $V$ of 0-dimensional simplices is called the vertex set of $\mathcal{K}$. The set given by the union of the simplices in $\mathcal{K}$ is called the space of the simplicial complex and is denoted $|\mathcal{K}|$. For each $\sigma \in |\mathcal{K}|$, there exists a unique simplex $K$ of $\mathcal{K}$ containing $\sigma$ in its interior. Define the barycentric coordinate function $\sigma \to V$ by letting $\sigma(v) = 0$ if $v$ is not a vertex of $K$ and otherwise by letting $\sigma(v)$ be the corresponding barycentric coordinate of $\sigma$ in the simplex $K$. For each vertex $v \in V$, the star of $v$, denoted $\mathrm{St}(v)$, is the set of $\sigma \in |\mathcal{K}|$ such that $\sigma(v) > 0$. The closed star of $v$, denoted $\mathrm{Cl}\,\mathrm{St}(v)$, is the closure of $\mathrm{St}(v)$.

A subdivision of a simplicial complex $\mathcal{K}$ is a simplicial complex $\mathcal{K}^*$ such that each simplex of $\mathcal{K}^*$ is contained in a simplex of $\mathcal{K}$ and each simplex of $\mathcal{K}$ is the union of simplices in $\mathcal{K}^*$. Obviously $|\mathcal{K}| = |\mathcal{K}^*|$. We need the following Theorem on simplicial subdivisions for our Approximation Theorem below (cf. Spanier, Chapter 3, for a proof).

**Theorem B.1.** *For every simplicial complex $\mathcal{K}$ and every positive number $\lambda > 0$, there exists a simplicial subdivision $\mathcal{K}^*$ such that the diameter of each simplex of $\mathcal{K}^*$ is at most $\lambda$.*

A multisimplex is a set of of the form $K_1 \times \cdots K_m$, where for each $i$, $K_i$ is a simplex. A multisimplicial complex $\mathcal{K}$ is a product $\mathcal{K}_1 \times \cdots \times \mathcal{K}_m$, where for each $i$, $\mathcal{K}_i$ is a simplicial complex. (The vertex set $V$ of a multisimplicial complex $\mathcal{K}$ is the set of all $(v_1, \ldots, v_m)$ for which for each $i$, $v_i$ is a vertex of $\mathcal{K}_i$. The space of the multisimplicial complex is $\prod_i |\mathcal{K}_i|$ and is denoted $|\mathcal{K}|$. For each vertex $v$ of $\mathcal{K}$, the star of $v$, $\mathrm{St}(v)$, is the set of all $\sigma \in |\mathcal{K}|$ such that for each $i$, $\sigma_i \in \mathrm{St}(v_i)$. The closure of this set is $\mathrm{Cl}\,\mathrm{St}(v)$. A subdivision of a multisimplicial complex $\mathcal{K}$ is a multisimplicial complex $\mathcal{K}^* = \prod_i \mathcal{K}_i^*$ where for each $i$, $\mathcal{K}_i^*$ is a subdivision of $\mathcal{K}_i$. In the following, $\mathcal{K}$ is a fixed multisimplicial complex and $\mathcal{L}$ is a fixed simplicial complex.

**Definition B.2.** A map $f : |\mathcal{K}| \to |\mathcal{L}|$ is called multisimplicial if for each multisimplex $K$ of $\mathcal{K}$ there exists a simplex $L$ in $\mathcal{L}$ such that:

1. $f$ maps each vertex of $K$ to a vertex of $L$;
2. $f$ is multilinear on $|K|$; i.e., for each $\sigma \in |K|$, $f(\sigma) = \sum_{v \in V} f(v) \times \prod_i \sigma_i(v_i)$.

By Property 1 of the Definition, vertices of $K$ are mapped to vertices of $L$. Therefore, for each $\sigma \in |K|$, $f(\sigma)$ is an average of the values at the vertices of $K$. Since the simplex $L$ is a convex set, the image of the multisimplex $K$ is contained in $L$. If $\mathcal{K}$ is a simplicial complex then Definition B.2 coincides with the usual definition of a simplicial map. In this case the image of a multisimplex $K$ under $f$ is a simplex of $L$, but in the multilinear case the image of $K$ could be a strict subset of $L$.

**Definition B.3.** Let $g : |\mathcal{K}| \to |\mathcal{L}|$ be a continuous map. A multisimplicial map $f : |\mathcal{K}| \to |\mathcal{L}|$ is a multisimplicial approximation to $f$ if for each vertex $\sigma \in |\mathcal{K}|$ there exists a simplex $L$ of $\mathcal{L}$ that contains both $f(\sigma)$ and $g(\sigma)$.

We could equivalently define a multisimplicial approximation by requiring that for each $\sigma$, if $g(\sigma)$ belongs to the interior of a simplex $L$ then $f(\sigma)$ belongs to the closure of $L$. Put differently, if $g(\sigma)$ belongs to $\mathrm{St}(w)$ then $f(\sigma)$ belongs to $\mathrm{Cl}\,\mathrm{St}(w)$. We are now state and prove a Multisimplicial Approximation Theorem.

**Theorem B.4.** *Suppose that $g : |\mathcal{K}| \to |\mathcal{L}|$ is a continuous map. Then there exists a subdivision $\mathcal{K}^*$ of $\mathcal{K}$ and a multisimplicial approximation $f : |\mathcal{K}^*| \to |\mathcal{L}|$ of $g$.*

*Proof.* The collection $\{g^{-1}(\mathrm{St}(w)) \mid w \text{ is a vertex of } \mathcal{L}\}$ is an open covering of $|\mathcal{K}|$. Let $\lambda > 0$ be a Lebesgue number of this covering; i.e., every subset of $|\mathcal{K}|$ whose diameter is less than $\lambda$ is included in some set of the collection. By Theorem B.1, there exists for each $i$ a simplicial subdivision $\mathcal{K}_i^*$ of $\mathcal{K}_i$ such that the diameter of each simplex is less than $\lambda/2$. Then for each vertex $v$ of $K^*$, $\mathrm{St}(v)$ has diameter less than $\lambda$. (Recall that we use the $\ell_\infty$ norm.) We first define a vertex map $f^0$ from the vertex set of $\mathcal{K}^*$ to the vertex set of $\mathcal{L}$ as follows. For each vertex $v$ of $\mathcal{K}^*$, since the diameter of $\mathrm{St}(v)$ is less than $\lambda$, there exists a vertex $w$ of $\mathcal{L}$ such that $g(\mathrm{St}(v)) \subset \mathrm{St}(w)$. Let $f^0(v) = w$. Suppose $v^1, \ldots, v^k$ are vertices of a multisimplex $K$. We claim that there exists a simplex $L$ of $\mathcal{L}$ whose vertex set includes $f^0(v^j)$ for all $j$. Indeed, since the $v^j$'s are vertices of a multisimplex, we have that $\cap_j \mathrm{St}(v^j)$ is nonempty. Therefore,

$$\emptyset \neq g(\cap_j \mathrm{St}(v^j)) \subseteq \cap_j g(\mathrm{St}(v^j)) \subseteq \cap_j \mathrm{St}(f^0(v^j)) .$$

Therefore, the vertices $f^0(v^j)$ span a simplex in $\mathcal{L}$. Since $f^0$ maps vertices of a multisimplex to vertices of a simplex, there exists a unique, well-defined multilinear extension of $f^0$, call it $f$. To finish the proof we must show that $f$ is a multisimplicial approximation of $g$. Let $\sigma$ be an interior point of a multisimplex $K$ and let $L$ be the simplex containing $g(\sigma)$ in its interior. For every vertex $v$ of $K$, we have by construction that $g(\mathrm{St}(v)) \subseteq \mathrm{St}(f(v))$. Thus $g(\sigma)$ belongs to the star of $f(v)$ for each vertex of $K$. In particular, the set of vertices $f(v)$ where $v$ is a vertex of $K$ span a subsimplex $L'$ of $L$. Since $f(\sigma)$ belongs to $L'$, $f$ is a multisimplicial approximation of $g$. $\square$

The proof of the Theorem shows a slightly stronger result. Let $\eta = \lambda/2$, where $\lambda$ is as defined in the proof. If each $\mathcal{K}_i^*$ is subdivision of $\mathcal{K}_i$ such that the diameter of each simplex is at most $\eta$ then $g$ admits a multisimplicial approximation $f : |\mathcal{K}^*| \to |\mathcal{L}|$. Thus, we have the following Corollary.

**Corollary B.5.** *There exists $\eta > 0$ such that for each subdivision $\mathcal{K}^*$ of $\mathcal{K}$ with the property that the diameter of each multisimplex is at most $\eta$, there exists a multisimplicial approximation $f : |\mathcal{K}^*| \to |\mathcal{L}|$ of $g$.*

B.2. **Construction of a Convex Map on a Polyhedral Subdivision.** This section describes the construction of a convex map associated with a polyhedral refinement of a simplicial subdivision.

Let $\mathcal{T}$ be a simplicial complex obtained from a simplicial subdivision of a compact convex subset $\Sigma$ of $\mathbb{R}^n$. Let $d$ be the dimension of $\Sigma$. The polyhedral complex $\mathcal{P}$ is derived from $\mathcal{T}$, as follows (Eaves and Lemke, 1981). For each simplex $\tau \in \mathcal{T}$ whose dimension is $d-1$, let $H_\tau = \{z \in \mathbb{R}^n \mid a'_\tau z = b_\tau\}$ be the hyperplane that includes $\tau$, and if $d < n$ is orthogonal to $\Sigma$. Then each closed $d$-dimensional admissible polyhedron of $\mathcal{P}$ has the form $\Sigma \cap [\cap_\tau H_\tau^{p_\tau}]$ where each $p_\tau \in \{+, -\}$ and $H_\tau^+$ and $H_\tau^-$ are the two closed halfspaces whose intersection is $H_\tau$. Enlarge $\mathcal{P}$ by applying the rule that each lower-dimensional polyhedral face of an admissible polyhedron is also admissible. By construction, the closure of each simplex in $\mathcal{T}$ is partitioned by admissible polyhedra of $\mathcal{P}$, any two nondisjoint admissible polyhedra meet in a common face that is also an admissible polyhedron, and each admissible polyhedron is convex. Associate with $\mathcal{P}$ the map $\gamma : \Sigma \to \mathbb{R}_+$ for which $\gamma(\sigma) = \sum_\tau |a'_\tau \sigma - b_\tau|$. Then $\gamma$ is convex and piecewise affine. Moreover, each maximal domain on which $\gamma$ is linear is an admissible polyhedron of $\mathcal{P}$.

## Appendix C. Example

Hauk and Hurkens (2002) study a game with an equilibrium component whose index is zero and for which the projection from a neighborhood in the equilibrium graph is surjective onto a neighborhood of the game. Thus every nearby game has nearby equilibria. Theorem 1.1 implies that this cannot remain true for every inflated game. Indeed they show that it fails when a single (carefully selected) redundant strategy is added. The following example seems to require at least two redundant strategies to be added.

Consider the 2-player game $G = (A; B)$ whose payoff matrices $A$ and $B$ for players 1 and 2 are shown below along with the labels for their pure strategies.

| | | | *Left L* | *Center C* | *Right R* |
|---|---|---|---|---|---|
| Player 1's payoff matrix $A$ | | | | | |
| *Outside Option* | $OO$ | : | 0 | 0 | 0 |
| *Top* | $T$ | : | 1 | $-3$ | $-3$ |
| *Middle* | $M$ | : | $-2$ | $-1$ | 1 |
| *Bottom* | $B$ | : | $-1$ | $-2$ | 1 |
| Player 2's payoff matrix $B$ | | | | | |
| *Outside Option* | $OO$ | : | 0 | 0 | 0 |
| *Top* | $T$ | : | 1 | 0 | 0 |
| *Middle* | $M$ | : | 0 | 1 | 0 |
| *Bottom* | $B$ | : | 0 | 0 | 1 |

The game $G$ has an equilibrium component $C$ whose index is zero. In $C$ player 1 uses only his outside option $OO$, and player 2 uses any mixed strategy in the convex polytope for which $OO$ is an optimal reply by player

1. The six vertices of the polytope, labeled $P1, \ldots, P6$ are:

| | $P1$ | $P2$ | $P3$ | $P4$ | $P5$ | $P6$ |
|---|---|---|---|---|---|---|
| $L:$ | 0.75 | 0.75 | 0.5 | 0.2 | 0 | 0 |
| $C:$ | 0.25 | 0 | 0 | 0.2 | 0.5 | 1 |
| $R:$ | 0 | 0.25 | 0.5 | 0.6 | 0.5 | 0 |

Numerical experiments indicate that every small perturbation of an inflated game obtained by adding a single redundant strategy for player 2 has equilibria (generically, two or four, their indices summing to zero) whose reduced forms are near $C$. However, the following perturbed inflated game has no equilibrium whose reduced form is near $C$. The inflation of $G$ adds two redundant strategies: (1) a convex combination of P1, P2, and Left in which nearly all the weight is given to P1, and (2) a convex combination of P1, P4, and P6 in which nearly all the weight is given to the mixture P* = (2/3)P1+(1/3)P6 and the remainder divided between P4 and P6. Note that (1) is outside $C$ and (2) is inside. The perturbation gives small bonuses to the two redundant strategies, with the first greater than the second. A particular choice of these weights and bonuses produces the following game that has no equilibrium near $C$:

| | | | *Left L* | *Center C* | *Right R* | (1) | (2) |
|---|---|---|---|---|---|---|---|
| Player 1's payoff matrix $A$ | | | | | | | |
| *Outside Option* | $OO$ | : | 0 | 0 | 0 | 0 | 0 |
| *Top* | $T$ | : | 1 | −3 | −3 | .01 | −1.012 |
| *Middle* | $M$ | : | −2 | −1 | 1 | −1.748 | −1.485 |
| *Bottom* | $B$ | : | −1 | −2 | 1 | −1.24 | 1.485 |
| Player 2's payoff matrix $B$ | | | | | | | |
| *Outside Option* | $OO$ | : | 0 | 0 | 0 | 0 | 0 |
| *Top* | $T$ | : | 1 | 0 | 0 | .7525 | .497 |
| *Middle* | $M$ | : | 0 | 1 | 0 | .245 | .497 |
| *Bottom* | $B$ | : | 0 | 0 | 1 | .0025 | .006 |

The redundant strategies (1) and (2) were obtained as follows. When a single redundant strategy is added and given a bonus, the graph of equilibria whose reduced forms are near $C$, over the simplex of possible choices of the redundant strategy, has two folds, one above the line between P1 and P2 and one above the line between P4 and P*. By adding *two* redundant strategies, each slightly *outside* the projection of a fold, one ensures that the perturbed inflated game has no equilibrium whose reduced form is near $C$. This procedure suggests an algorithm, but unfortunately the proof of Theorem 1.1 provides no hint about its general formulation.

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