The Ethics of Intergenerational Risk

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This paper re-examines the evaluation of intergenerational allocations in an uncertain world. It axiomatically characterizes a class of criteria that avoid serious drawbacks of expected discounted utilitarianism concerning the choice of the discount factor and the insensitivity to distributional issues. The distinctive feature of the proposed criteria is to assess alternatives based on specific information about the evolution of technology, the intensity and timing of resolution of risk, and the scarcity of resources.

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1. Introduction

1.1. Motivation

Human activities today impact the welfare of future generations. Some activities, as investing in new technologies, are likely to improve future living conditions. Others, as those inducing climate change, are likely to worsen them. To provide sensitive policy recommendations, society needs to aggregate the conflicting interests of present and future generations based on well-defined normative principles.

In economics, the most prominent welfare criterion to assess intergenerational risks is expected discounted utilitarianism (EDU). Social welfare is the discounted sum across time and states of nature of the well-being of each generation at each state of nature. Despite its intuitive formulation, the EDU criterion presents at least two major weaknesses, which motivate this study.

First, discounting discriminates against later generations. From an ethical viewpoint, discounting is "a practice which is ethically indefensible and arises merely from the weakness of the imagination" (Ramsey, 1928, p.543). From a practical viewpoint, however, discounting can be justified by the probability of extinction (Dasgupta and Heal, 1979; Stern, 2007).[1] More importantly, discounting reduces the excessive moral obligations to sacrifice the welfare of earlier generations for the benefit of better-off later generations (Zuber and Asheim, 2012). Yet, the choice of the "correct" discount factor remains elusive and constitutes an unsolved moral dilemma for practitioners and policymakers.

Second, the EDU criterion is insensitive to permutations of generations' well-being across time and states of nature.[2] This property is directly inherited from its static counterpart, expected utilitarianism, and has been largely discussed in relation to Harsanyi (1955)'s characterization (Adler and Sanchirico (2006); Epstein and Segal (1992); Fleurbaey (2010); Gajdos and Maurin (2004)). In some situations, society might prefer giving equal chances to both individuals rather than assigning a prize to one individual only, as advocated by Diamond (1967) and Epstein and Segal (1992). In other situations, instead, society might prefer assigning a prize to both individuals with 50% probability, rather than assigning the prize for sure, but randomizing on the recipient, as advocated by Broome (1984) and Fleurbaey (2010). As these concerns for "ex-ante egalitarianism" and "ex-post fairness" are both normatively appealing but appear to be mutually inconsistent (see Fleurbaey (2010)), EDU continues to be the prevalent criterion in applications.[3]

In this paper, I explore an approach to intergenerational ethics that proves able to overcome these weaknesses. The main innovation is to let some specific information guide society in the evaluation of alternative distributions of resources. More precisely, information related to the resource distribution problem—i.e. the evolution of technology, the intensity and timing of resolution of risk, and the scarcity of resources—allow formulating more versatile principles of intergenerational justice. These principles do not alone determine how society should discount the future or how generations' assignment should be aggregated. Rather, these principles establish how society should use the information available to determine both the social discount rate and how to measure and aggregate each generation's well-being.

This explicit relationship between the ranking of allocations and the resource distribution problem is new to the literature on intergenerational risk. The standard approach is to define "universal" criteria. These criteria evaluate allocations independently of the problem and are, thus, simple and analytically tractable. Yet, they seem unable to provide appealing policy recommendations for each problem, as discussed above for the EDU

[1] Dasgupta and Heal (1979, p. 262) clarify "that one might find it ethically reasonable to discount future utilities at positive rates ... because there is a positive chance that future generations will not exist."

[2] Formally, permuting the (discounted) well-being across time and (equally likely) states of nature leaves social welfare unchanged. Crucially, this conclusion does not rest on what index of well-being is used to evaluate the quality of life of a generation at a specific state of nature. The insensitivity to permutations holds whether society uses each generation's cardinal utility or a (concave) transformation of it or a direct evaluation of generation's assignment.

[3] Alternatives to EDU have been recently proposed in the literature. Adler and Treich (2014) propose a prioritarian social welfare function, which aggregates concave transformations of each generation's expected utility. Fleurbaey and Zuber (2015), instead, propose to first evaluate the welfare at each state of nature through the "equally distributed equivalent" and than aggregate such indexes over states of nature.

criterion. A different approach is to associate a subset of optimal allocations to each problem, as in the fair allocation theory literature (see the recent survey by Thomson (2011)). This approach is flexible and provides policy recommendations tailored to the specific problems faced by society. The drawback is that "optimal" allocations are often of little help in second best situations, where fine-grained welfare criteria are more appropriate. In this paper, I integrate the two approaches. This *hybrid* approach combines the simplicity and analytical tractability of fine-grained welfare criteria with the flexibility of choosing optimal allocations.[4]

The criterion characterized here can be seen as adding an endogenous "equivalent scale" to EDU. The equivalent scale tells that the legitimate claim to consumption of each generation in each state should depend on what the cost for society is for providing consumption in that specific time and state. To illustrate, consider a stylized two-period model. In the second period, a small asteroid hits the earth with probability π and, as a result, the productivity of technology is drammatically reduced (without having any direct effect on population). Standard EDU implicitly assumes that each generation in each state deserves the same consumption level and—by concavity of u—it penalizes allocations that deviate from this *strongly egalitarian* ideal. Yet, since less is available for consumption should the asteroid hit, this strongly egalitarian allocation is either feasible and inefficient or unfeasible. The allocation that maximizes EDU is instead efficient. Thus, when searching for the optima, society attempts to fight inequalities that are inevitable, leading to large sacrifices for the present generation which are difficult to justify morally.

For instance, at the limit for u being infinitely concave, society is generally viewed as extremely egalitarian. Yet, the optimal allocation is strikingly *non* egalitarian. The consumption of the first-period generation equals that of the second-period generation in the state in which the asteroid hits; should the asteroid not hit, the second-period generation would enjoy a consumption that could be arbitrarily larger than that of the first-period generation. What is particularly non-convincing is that society disregards that the second-period generation happens to live in the asteroid-hit scenario only with probability π, which can be arbitrarily close to 0.

The introduction of equivalent scales avoids these issues. Society penalizes allocations that deviate from a *weakly egalitarian* and *efficient* reference allocation. It is weakly egalitarian as the reference assigns to each generation equally desirable consumption lotteries. Moreover, since it is efficient, it reflects the technology that is available in each state. Since the asteroid's hit reduces productivity, the second-period generation cannot legitimately claim the same consumption as if the asteroid didn't hit. Of course, society should compensate the second-period generation for such a risk, but only sofar as this generation is considered worse-off than the previous generation.

In the next section, I introduce a two-period model. This simple example allows me: to illustrate the main equity principles; to present the class of *reference-dependent utili-*

[4]For a similar approach dealing with the aggregation of individuals' preferences, see Fleurbaey and Maniquet (2011) and, for the intergenerational setting, Fleurbaey and Zuber (2014).

tarian welfare criteria; and to compare these criteria with EDU. In Section 3, I formalize the model and the axioms and derive the main results. Before this, I shortly place the contribution within the literature.

1.2. Related literature

The standard approach to the social evaluation of intergenerational risks is to rely on Harsanyi (1955)'s characterization of expected utilitarianism, where individuals are reinterpreted as generations. Arguably, however, Harsanyi's setting is not the most appropriate to address long-term intergenerational risk. First, it is arbitrary to assume that generations have preferences over their contingent consumptions as, by the time they will be born, most of the risk might be resolved. Thus, the problem cannot be to aggregate generations' preferences, as these preferences do not exist to start with. Second, risk resolves gradually over time and not in "one shot." This implies that generations face different risks and cannot be treated anonymously with respect to the time they live in. Finally, extinction is considered a fundamental justification for intergenerational discounting, but has no place in Harsanyi's framework. To address these issues, I adopt a model of gradual resolution of risk similar to Kreps and Porteus (1978) with two main changes. First, I allow for the possibility of extinction. Second, I assume that each generation t is born after the risk at t is resolved: all risk is then entirely borne by society, as in Asheim and Brekke (2002) and, more recently, Asheim and Zuber (2016).[5]

The analysis of individual preferences in the context of intertemporal models with gradual resolution of risk includes Kreps and Porteus (1978), Epstein and Zin (1989), Weil (1990), and Traeger (2012). As in Kreps and Porteus (1978), we highlight that the timing of resolution of risk matters for the ranking of allocations. As in Epstein and Zin (1989) and Weil (1990), we obtain (more tractable) isolelastic functional forms. Traeger (2012) introduces the concept of "intertemporal risk aversion" to model an individual that dislikes lotteries with persistent outcomes; this requirement, however, may well capture individual's attitude towards risk and time, but it seems far less appealing for social preferences as it directly violates the social concern for ex-post fairness.

A more fundamental difference emerges in terms of time consistency, i.e. the requirement that the ranking of two allocations with a common first period assignment is unchanged when this assignment is realized. The *reference-dependent utilitarian* criteria are time inconsistent.[6] This is the "cost" the proposed approach pays to address the above-mentioned drawbacks of EDU. However, this cost might not be considered very large. First, time inconsistency is not in conflict with "rational" social preferences: when computing today's distributive policy, society needs to be sophisticated and take into account how this policy

[5] Asheim and Zuber (2016) builds on recent advances in the utility-streams literature on intergenerational justice, and in particular on the rank-discounted utilitarian criterion (Zuber and Asheim, 2012), and studies how to rank social situations in which each potential individual is characterized by a utility level and a probability of existence.

[6] Except for the degenerate case for which the criteria do not depend on the specific resource distribution problem faced by society.

will influence future optimal policies (Pollak, 1968). Second and more importantly, time inconsistency is unavoidable in the intergenerational setting—even without risk—as it is a direct implication of basic principles of intergenerational equity (Asheim and Mitra, 2016).

Reference-dependent utilitarianism also draws a parallel with reference-dependent preferences (see Koszegi and Rabin (2006) and Ok et al. (2015)). These preferences depend not only on the assigned alternative, but also on the reference point adopted for the evaluation. In behavioral economic models, the reference is set to the status quo or to match individual's expectations. In the present setting, the reference is given a normative justification and is singled out as the unique allocation that is both efficient and equitable, in the sense of Asheim and Brekke (2002).

Finally, our approach is also related to Dhillon and Mertens (1999)'s alternative to expected utilitarianism. Dhillon and Mertens suggest additively aggregating normalized von Neumann-Morgenstern utility functions, where each individual's utility is set to have infimum 0 and supremum 1 on a set of admissible prospects. They suggest the admissible prospects to be "limited only by feasibility and justice" (1999, p. 476), but do not specify how. A distinguishing feature of the present contribution is to axiomatically formalize how the welfare criterion should depend on the problem faced by society.

2. A two-period example

To illustrate the *reference-dependent utilitarian* criteria, consider the following class of two-period risky intergenerational problems (similar to Selden (1978)). At period 0, an amount $\omega > 0$ of a good is available. This can be partly allocated for consumption of generation 0, say x_0, and, for the remaining part, invested in capital, say k_1. When taking decisions at 0, society has a probabilistic knowledge about the future. First, extinction can arise with positive probability: let $\pi \in (0, 1]$ be the likelihood that generation 1 exists. Second, the output available in period 1, i.e. Ak_1, depends on the realization of the productivity shock A, which is a positive random variable with finite mean.[7] In period 1, a specific level of productivity realizes, say a. The output available is then ak_1, which can be allocated for the consumption of generation 1, say x_1^a. I denote the contingent consumption of generation 1 by x_1: it is a mapping that associates a consumption x_1^a to each possible realization a of the productivity shock A. A risky intergenerational problem is then identified by the endowment ω, by the survival probability π, and by the distribution of the productivity shock A.

A key feature of this class of problems is that intergenerational risk unfolds at different times. Some decisions—i.e. the consumption and investment choices at period 0—are taken without knowing their exact effect on future generations; other decisions—the con-

[7] The characterization result is developed in a setting where the number of states of nature is finite. This is not without loss of generality. With infinite many states of nature and arbitrary distributions of risk, the reference allocation exists only on a subset of ethical parameters. Yet, provided that a reference exists, the characterization of the criterion extends to continuous state spaces. I further discuss the existence of the reference in Fn.8.

sumption choices at period 1—are "more informed" as they can depend on the realization of the technology shocks. This has two main implications. First, generations are subject to different risks. Second, unless society is willing to waste resources when more turns out to be available, risk makes intergenerational inequalities unavoidable.

2.1. The reference and its role

To capture these unavoidable inequalities, I introduce the **reference** (allocation). The reference answers the following question:

> How would an egalitarian society distribute resources across generations?

The answer is here given by the unique allocation that satisfies the following two principles:

- *Efficiency.* An allocation cannot be the reference if there exists another feasible allocation that assigns at least as much consumption to each generation at each state of nature and strictly more to some.
- *Recursive equity.* At the reference, the consumption assigned to each generation at each state of nature is the certainty equivalent of the consumption lottery assigned to any later generations at states of nature that can still occur.

By *efficiency*, the reference is an allocation (r_0, r_1) such that $r_1 = A(\omega - r_0)$. By *recursive equity*, r_0 is the certainty equivalent of r_1; formally, society has to choose a real-valued function μ such that $r_0 = \mu^{-1} \circ E[\mu(r_1)]$. It is natural to select μ to be concave. The concavity of μ ensures that the consumption risk faced by later generations be compensated by a larger mean; the larger the concavity, the larger this compensation.[8]

If society was egalitarian, the optimal policy would be identified by the reference (r_0, r_1). In general, however, society might consider deviating from this reference. For example, a small reduction of the consumption of generation 0, if combined with a sufficiently large increase of consumption in period 1, might be considered welfare improving.

For a non-egalitarian society, the natural role of the reference is that of a watershed. Assume generation 0 is assigned a consumption that is smaller than at the reference, i.e. $x_0 < r_0$, while generation 1 is assigned a larger consumption than at the reference (in

[8] When μ is not too concave ($exp[\mu]$ is convex or μ is exponentially convex), the reference exists independently of the distribution of risk. To illustrate, assume $\mu(x) = \frac{x^{1-\theta}}{1-\theta}$ with $\theta > 0$; then, *recursive equity* and *efficiency* are satisfied when r_0 equalizes $(\omega - r_0)\left(E\left[A^{1-\theta}\right]\right)^{\frac{1}{1-\theta}}$. Assume the severity of damages A^{-1}, i.e. the inverse of the productivity, is Pareto distributed (thus fat-tailed) with tail index $\tau > 0$; formally, $\Pr(\bar{x} > x) = \left(\frac{a_{\max}^{-1}}{x}\right)^{\tau}$ for each $x \geq a_{\max}^{-1}$, where x is the inverse of the productivity shock and $a_{\max}$ is the largest value that the productivity shock can take. Then, $\left(E\left[A^{1-\theta}\right]\right)^{\frac{1}{1-\theta}}$ is finite only if $\theta < \tau + 1$. Consequently, $\theta \leq 1$ emerges as a sufficient condition for the existence of the reference. When instead $\theta \geq \tau + 1$, society is so concerned with the risk faced by generation 1 that, independently of r_0, the certainty equivalent of r_1 is arbitrarily close to 0 (but numerically undefined). The concavity restriction on μ does not emerge in the axiomatic characterization due to the assumption of a finite state space. Remark, that society is allowed to set a different function μ for each risky intergenerational problem or, if it satisfies the restriction above, fix a unique μ.

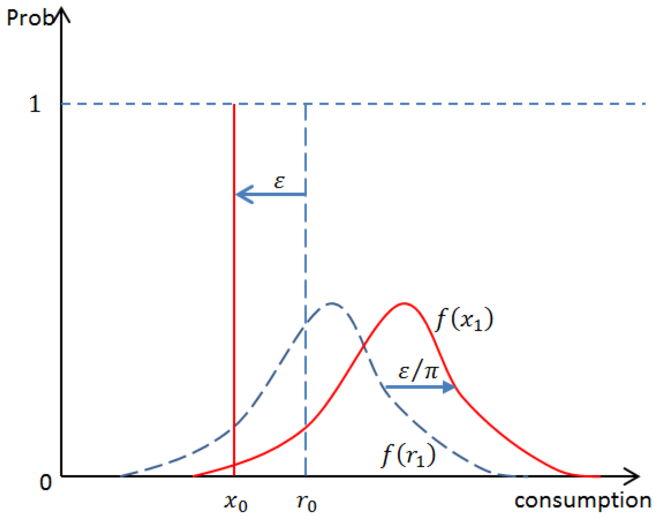

Figure 1: *Ex-ante equity* tells that (x_0, x_1) is socially less desirable than (r_0, r_1).

the sense of statewise dominance, i.e. larger in each state of nature), denoted $x_1 \gg r_1$. Then, society should consider generation 0 worse off than generation 1: there is an *ex-ante inequality* between the assignment of generations 0 and 1. Assume also that the expected consumption assigned to the generations is unchanged, i.e. there exists $\varepsilon > 0$ such that $x_0 = r_0 - \varepsilon$ and $x_1 = r_1 + \frac{\varepsilon}{\pi}$ (the division by the probability of existence π accounts for the fact that generation 1 might not exist). These allocations are represented in Fig.1, where $f(r_1)$ and $f(x_1)$ are the probability densities functions of r_1 and x_1 respectively, while r_0 and x_1 are certain and have probability mass of 1. Then, (x_0, x_1) can be thought of as obtained from (r_0, r_1) by transferring a (certain) consumption from generation 0 to generation 1. A society that is averse to ex-ante inequalities—or satisfies *ex-ante equity*—cannot prefer (x_0, x_1) to the reference (r_0, r_1).

The reference also guides society in the evaluation of *ex-post inequalities*. Assume generation 0 is assigned the reference consumption $x_0 = r_0$. Generation 1 is instead assigned a consumption x_1, which happens to be a mean-preserving spread of the reference consumption r_1: in some states of nature, generation 1 is assigned more than at the reference; in others, it is assigned less than at the reference. This construction is represented in Fig.2. Thus, independently of which state of nature arises, the allocation (x_0, x_1) implies a larger probability of intergenerational inequality than at the reference. A society that is averse to ex-post inequalities—or satisfies *ex-post equity*—cannot prefer (x_0, x_1) to the reference (r_0, r_1).

2.2. The reference-dependent utilitarian criterion

In Section 3, I combine *ex-ante* and *ex-post equity* with other axioms and characterize the *reference-dependent utilitarian* criterion. These other axioms are common in the literature: "monotonicity" requires that social welfare increases when assigning more consumption;

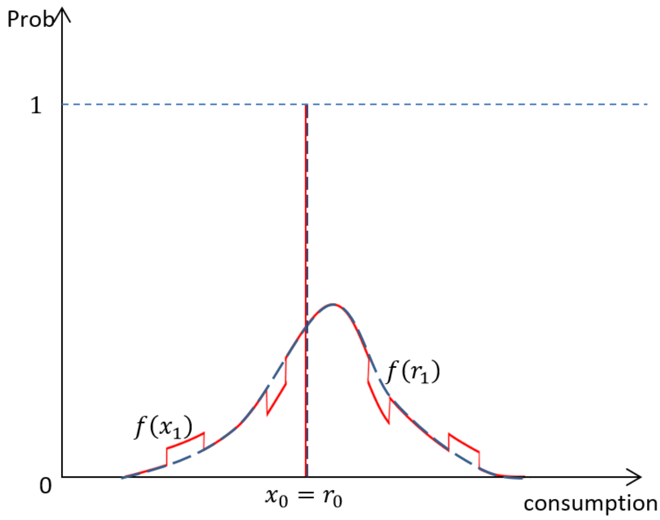

Figure 2: *Ex-post equity* tells that (x_0, x_1) is socially less desirable than (r_0, r_1).

"continuity" says that small changes of the allocation should not cause large jumps in the level of social welfare; two types of "separability" give an additive structure to the representation; and "ratio-scale comparability" requires the ranking to be invariant to rescaling allocations. In the remaining part of this section, I illustrate and discuss the *reference-dependent utilitarian* criterion and contrast it to the EDU criterion.

Given the reference $r \equiv (r_0, r_1)$, the *reference-dependent utilitarian* welfare at allocation $x \equiv (x_0, x_1)$ is measured by:

$$W(x; r) = \underbrace{w\left(\frac{x_0}{r_0}\right)}_{\text{welfare of gen. 0}} + \underbrace{\frac{E[r_1]}{r_0}\pi}_{\substack{\text{risk-adjusted} \\ \text{discount factor}}} \cdot \underbrace{w \circ v^{-1}\left(\frac{E\left[r_1 v\left(\frac{x_1}{r_1}\right)\right]}{E[r_1]}\right)}_{\text{welfare of gen. 1}}, \tag{1}$$

where $w(z) = \dfrac{z^{1-\rho}}{1-\rho}$ and $v(z) = z^{1-\gamma}$ with $\rho, \gamma \geq 0$ (the ethical interpretation of these parameters is discussed in the following).

Society evaluates the consumption assigned to each generation in relation to what this generation would be assigned at the reference. To understand its implications, I discuss how social welfare changes around the reference. When each generation is assigned her reference consumption, i.e. $x = r$, each achieves the same well-being and social welfare is $W(r; r) = \frac{1}{1-\rho} + (E_\pi[r_1]/r_0)\,\pi\frac{1}{1-\rho}$.

Consider a transfer of consumption $\varepsilon > 0$ from generation 0 to generation 1. At the margin, social welfare is unchanged:[9]

$$\lim_{\varepsilon \to 0} \frac{W\left(\left(r_0 - \varepsilon, r_1 + \frac{\varepsilon}{\pi}\right); r\right)}{\varepsilon} = W(r; r).$$

At the reference r, the marginal rate of substitution between the expected consumption of the two generations is 1, ensuring that no generation is discriminated against.

The **risk-adjusted discount factor** defined in (1) is crucial. The risk-adjusted discount factor expresses the weight that society gives to generation 1 in the social welfare function. Two forces govern discounting. The first one is due to the gradual resolution of risk. As later generations face more consumption risk than earlier ones, their assignment at the egalitarian reference consists of a larger expected consumption $E[r_1] \geq r_0$ (by the concavity of the function μ). This leads to attributing larger weights to future generations the larger the uncertainty about the future. The second force, moving in the opposite direction, is due to the probability of extinction. When the extinction probability is positive, i.e. $\pi < 1$, saving of resources is more costly as it is not sure future generations can benefit from it. This leads to attributing a smaller weight to future generations. Depending on which force prevails, the discount factor can be above or below 1.

The above-defined transfer across generations (weakly) reduces social welfare as soon as ε is not marginal. Intuitively, by *ex-ante equity,* society is averse to redistributions of resources across generations that increase the gap with the reference. This aversion is measured by the **ex-ante inequality-aversion parameter** ρ. When $\rho = 0$, society is indifferent to ex-ante inequalities and any such transfer leaves social welfare unchanged. The larger ρ, the more society is reluctant to redistribute resources across generations away from the reference. At the limit for $\rho \to \infty$, redistribution of resources across generations is socially unacceptable.

Now, starting again from the reference, consider a mean-preserving spread in generation 1's assignment. The new allocation is $(r_0, r_1 + \varepsilon z)$, where z is a zero-mean noise term and $\varepsilon > 0$ is a scalar such that $r_1 + \varepsilon z \gg 0$. At the margin, social welfare is unchanged:

$$\lim_{\varepsilon \to 0} \frac{W\left((r_0, r_1 + \varepsilon z); r\right)}{\varepsilon} = W(r; r).$$

At the reference r, the marginal rate of substitution between the probability-weighted consumption assigned to generation 1 at two different states of nature is 1. This ensures that the criterion does not discriminate between states of nature.

When instead the noise term is not marginal, the mean-preserving spread (weakly) reduces social welfare. Intuitively, by *ex-post equity,* society is averse to redistributions of resources across states of nature: such redistribution widens the inequalities across generations which will eventually arise, independently of the state of nature. This aversion is measured by the **ex-post inequality-aversion parameter** γ. When $\gamma = 0$, society

[9] Note that the transfer is weighted by the probability of extinction to ensure that, in expected terms, the same consumption is distributed.

is indifferent to such inequalities and the mean-preserving spread leaves social welfare unchanged. The larger γ, the more society is reluctant to redistribute resources across states of nature away from the reference. At the limit for $\gamma \to \infty$, redistribution of resources across states of nature is socially unacceptable.

Remark that the ex-ante and ex-post inequality-aversion parameters are independent of the choice of the reference (and thus of the function μ) and reflect different value judgements. No matter which reference is chosen, these parameters measure the willingness of society to deviate from it: the ex-ante parameter across time; the ex-post parameter across states.

2.3. The optimal distribution of resources

I next discuss the implications of the *reference-dependent utilitarian* criterion for optimality by comparing its first order condition with that of the EDU criterion. A society that maximizes (1) would optimally select an allocation $x^* \equiv (x_0^*, x_1^*)$ such that $x_1^* = A(\omega - x_0^*)$ and:[10]

$$w'\left(\frac{x_0^*}{r_0}\right) = E[A]\,\pi w'\left(\frac{E[x_1^*]}{E[r_1]}\right). \tag{2}$$

In contrast, an expected utilitarian planner would optimally select an allocation $\hat{x} \equiv (\hat{x}_0, \hat{x}_1)$ such that $\hat{x}_1 = A(\omega - \hat{x}_0)$ and:

$$u'(\hat{x}_0) = \beta E\left[Au'(\hat{x}_1)\right]. \tag{3}$$

The no-risk case. With probability 1, the productivity parameter is $a > 0$. Then, the reference r assigns the same consumption $r_0 = r_1 = a(\omega - r_0)$ to both generations. Thus, the first order condition (2) simplifies to:

$$w'(x_0^*) = a\pi w'(x_1^*), \tag{4}$$

which is equivalent to (3) when $\beta = \pi$ and $u(z) = \frac{z^{1-\eta}}{1-\eta}$ with $\eta = \rho$. The first order condition (4) is a natural requirement for the optimal allocation. Society is willing to give more (less) consumption to generation 1 with respect to generation 0 if, for each additional unit of consumption saved in period 0, more (less) than one unit can be expected to become available to generation 1. Thus, $a\pi > 1$ implies that $x_1^* > x_0^*$ and, similarly, $a\pi < 1$ implies that $x_1^* < x_0^*$.

The risky case. With risk, the two first order conditions diverge. To simplify the comparison, I assume that the EDU discount factor corresponds to the survival probability, i.e. $\beta = \pi$, and that the EDU evaluation function is $u(z) = \frac{z^{1-\eta}}{1-\eta}$. Then, (2) can be written

[10] Remark that the reference (r_0, r_1) is here fixed. By *efficiency* and *recursive equity,* it satisfies $r_1 = A(\omega - r_0)$ and $r_0 = \mu^{-1} \circ E[\mu(r_1)]$, where μ is concave. In the presence of fat-tailed catastrophic risks, concavity of μ does not ensure the existence of the reference: a sufficient condition is that μ is exponentially convex, i.e. not too concave.

as:

$$\left(\frac{x_0^*}{r_0}\right)^{-\rho} = E\left[A\right]\pi\left(\frac{\omega - x_0^*}{\omega - r_0}\right)^{-\rho}, \tag{5}$$

whereas (3) as:

$$(\hat{x}_0)^{-\eta} = E\left[A^{1-\eta}\right]\pi\left(\omega - \hat{x}_0\right)^{-\eta}. \tag{6}$$

As before, the optimality condition for the *reference-dependent utilitarian* criterion (5) requires assigning to generation 0 a smaller (larger) consumption than at the reference if, for each unit of consumption saved in period 0, more (less) than one unit is expected to become available in period 1. Thus, $E\left[A\right]\pi > 1$ implies that $x_0^* < r_0$ and $x_1^* \gg r_1$. Symmetrically, $E\left[A\right]\pi < 1$ implies that $x_0^* > r_0$ and $x_1^* \ll r_1$. Intuitively, the simple structure of the problem limits the choice of society: if the optimal allocation is non-wasteful, society can only redistribute resources over time and not over states of nature. Thus, it seems natural to redistribute resources away from the egalitarian reference to the period in which resources provide the largest expected consumption.

Importantly, this does not mean that the *reference-dependent utilitarian* criterion is insensitive to risk. Compare two risky intergenerational problems that only differ in terms of the dispersion of the productivity shock, say A' has more weight in the tails than A. While $E\left[A\right] = E\left[A'\right]$, society accounts for the larger risk faced by generation 1 through the reference. As the dispersion of the productivity shock increases, an egalitarian society would assign a larger consumption to generation 1, implying $r_0' < r_0$ and $r_1' \gg r_1$. At unchanged assignment, generation 1 is now considered worse off than generation 0. By the concavity of w ($\rho \geq 0$), this triggers a consumption adjustment in favor of generation 1.

Consider now the optimality condition (6) for the EDU criterion. The crucial aspect is that technological risk is accounted for through the term $E\left[A^{1-\eta}\right]$, i.e. the $1-\eta$ moment of the productivity shock A. When $\eta = 0$, society selects the optimum based on the average productivity shock $E\left[A\right]$. As η increases, society becomes more and more concerned with small realizations of the shock. It follows that when society is sufficiently concerned with intergenerational inequalities, i.e. $\eta > 1$, the term $E\left[A^{1-\eta}\right]$ becomes very sensitive to the distribution of the technological shock A. In particular, when the probability of small productivity shocks is sufficiently high (the right tail of A^{-1} has more mass than the exponential distribution), the $1 - \eta$ moment of A might be infinite and (6) has no solution. In the literature, this puzzling result has been named the "dismal theorem" (see Weitzman (2009)). In such a case, no matter how small (but positive) x_0 and no matter how large $E\left[x_1\right]$, the EDU criterion always recommends saving more resources for the benefit of generation 1. As Weitzman explains: since "it cannot be the case that society would pay an infinite amount to abate one unit of carbon,...something must be very wrong in the formulation of the underlying model" (Weitzman, 2014, p.545). In response, some economists argue that the EDU criterion is not appropriate to evaluate intergenerational situations with fat-tailed risks (see Millner (2013) for an overview). Unfortunately, this leaves no policy guidance for the excluded class of situations, the relevance of which is ultimately an empirical question.[11] Other economists, instead, take this as evidence that society should not place too much importance to inequalities across time and states of nature, i.e. avoiding the problematic case of $\eta > 1$.[12] Normatively, however, it is (at least) contentious whether an egalitarian society would address inequalities by compressing the consumption of the "worse-off" present generation for the benefit of a "better-off" next period generation.

Surprisingly, this problem never emerges with the *reference-dependent utilitarian* criterion. Since $E[A]$ is finite, the optimal allocation selected by the FOC (5) always exists, independently of the distribution of the productivity shock A and of the inequality aversion parameters.

2.4. The stochastic social discount rate

The social discount rate is the typical measure adopted in the economic literature to describe the importance today of a unit of expected consumption tomorrow. The social discount factor expresses the trade-off between the marginal change in a future period and the marginal change at period 0 that leaves social welfare unchanged. Similarly to Traeger (2014), I consider here a stochastic version of the social discount rate, where a reduction of consumption assigned at period 0, i.e. dx_0, allows investing a fraction $d\varepsilon$ in a project with stochastic return A. For computational simplicity, I assume that the return on the investment A and the growth rate at x are jointly log-normally distributed.[13]

First, consider the case where x is a non-wasteful distribution of resources and, thus, a candidate for the optimal allocation in our two period example.[14] Then, the social discount rate for the *reference-dependent utilitarian* criterion is:

$$d(x;r) = \delta + \alpha + \rho(\bar{g}_x - \bar{g}_r), \tag{7}$$

where $\delta \equiv -\ln \pi$ is the rate of pure time preference, $\alpha \equiv -\ln E[A]$ is the expected return

[11] A number of recent contributions have highlighted that scientific knowledge about future climate sensitivity is well-described by fat-tailed distributions of risk (see, among others, Roe and Baker (2007); Weitzman (2009)). Recently, Martin and Pindyck (2015) study how to deal with interconnected threats of catastrophes: they identify which catastrophe should be averted first to minimize potential damages. In contrast, I question the ethics of intergenerational resource distributions in the presence of potentially catastrophic risks.

[12] By limiting the concavity of the evaluation function adopted to transform consumptions into an index of welfare, society assigns a significantly lower weight to states of nature with tight feasibility constraint (regardless of their probability) and, this way, limits the willingness of society to transfer resources to generations facing catastrophic risks. Then, society disregards rare events, no matter how catastrophic these might be. In contrast, Dasgupta (2008) has suggested moral arguments for society to select a log-concave evaluation function. Building on Chichilnisky (2000), Chichilnisky (2010) avoids this dilemma by characterizing subjective probabilities with a non-standard topological approach. These results provide a better match with recent neurological findings on individuals' behavior in the presence of catastrophes.

[13] See the online appendix for the derivation of the results presented in this subsection.

[14] Since x is a non-wasteful distribution of resources, $x_1 = A(\omega - x_0)$ and the return on the investment A and the growth rate at x have perfect positive correlation. In this case, the formula for the stochastic social discount rate depends on the distribution of A only through the reference.

of the stochastic project, and $\bar{g}_x$ and $\bar{g}_r$ are, respectively, the expected consumption growth rate at allocation x and at the reference r. The social trade-off between consumption at different periods is the sum of three terms. The first term reflects the probability of extinction: the higher the probability π that generation 1 exists, the smaller the rate of pure time preference δ. The second term reflects the expected return of the stochastic project (in the literature, it is usually set to 0 by assuming that $E[A] = 1$). The third term is the product between the ex-ante inequality aversion parameter ρ and the difference in expected growth between the allocation x and the reference r. If the growth rate at x is larger than at r, i.e. $(\bar{g}_x - \bar{g}_r) > 0$, generation 1 is assigned a larger consumption than at the reference, while generation 0 is assigned a smaller consumption than at the reference. This justifies discounting the consumption of generation 1 at a higher rate. The larger the difference between the growth rates, the larger the priority society attributes to the worse-off generation. The degree to which society reacts to such difference is measured by the ex-ante inequality aversion parameter ρ.

In the general case, the social discount rate becomes:

$$d(x;r) = \delta + \alpha + \rho(\bar{g}_x - \bar{g}_r) + \rho\left(\frac{\sigma_x^2}{2} - \frac{\sigma_r^2}{2}\right) - \gamma(1+\rho)\left(\frac{\sigma_x^2}{2} + \frac{\sigma_r^2}{2} - c\sigma_r\sigma_x\right),$$

where σ_x and σ_r are, respectively, the standard deviations of the growth rates at allocation x and at the reference r and c is their correlation. Thus, two further terms are added. The first term tells that if the variance of the growth rate at allocation x is larger than the variance of the growth rate at the reference r, society reacts by discounting the future more. The degree to which society reacts to the difference in the growth variances is given by the ex-ante inequality aversion parameter ρ. This force is contrasted by the second term, expressing the ex-post concern of society (and its interaction for the ex-ante concern) for having a larger risk of growth. In fact, society reduces the social discount rate in proportion to the variance of the difference between the growth rates, i.e. $\sigma_x^2 + \sigma_r^2 - 2c\sigma_r\sigma_x$. The degree to which society reacts to this variance is given by the sum of the ex-post inequality aversion parameter γ and the product of the ex-ante and ex-post inequality aversion parameters $\rho\gamma$.

The social discount rate for the EDU criterion is similar, but differences are crucial:

$$\bar{d}(x\quad) = \bar{\delta} + \alpha + \eta(\bar{g}_x\qquad) + \eta\left(\frac{\sigma_x^2}{2}\qquad\right) - \eta(1+\eta)\left(\frac{\sigma_x^2}{2}\qquad - c\sigma_r\sigma_x\right),$$

where $\bar{\delta} \equiv -\ln\beta$ is the rate of pure time preference. Two changes emerge. First, the parameters ρ and γ are substituted by the intertemporal elasticity η to consumption changes. Second and most importantly, the expectation and the variance of the growth rate at r disappear. Note that these terms all reduce the social discount rate of the reference-dependent utilitarian criterion as compared to the EDU criterion. The intuition is that the unavoidable risks, accounted for by r, justify some of the growth and volatility of x and increase the weight that society attributes to generation 1. This difference addresses the difficulty of the EDU criterion to attribute a significant social concern to future technological risk (see Nordhaus (2008); Traeger (2014).

3. The characterization result

3.1. A risky intergenerational problem[15]

Time is discrete and the horizon finite: $T \equiv \{0, ..., \bar{t}\}$, with $\bar{t} \geq 2$. At period 0, a stock of capital $k_0 > 0$ is available. Production takes place and transforms capital into output. Let Φ denote the set of all production functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ that are strictly increasing, continuous, and satisfy $\phi(0) = 0$. The output can be partly assigned for consumption of the current one-period living generation or, for the remaining part, saved for use in later periods.[16]

Later periods are characterized by two types of risk. First, extinction may arise before the end-period $\bar{t}$. Second, technology is randomly selected from Φ. To clarify, at each period, the decision about how to share output between consumption and investment is made without knowing whether later generations will exist and, conditional on existence, without knowing what technology will be available.

To formalize risk and its resolution over time, information disclosure takes the form of an event tree. An **event tree** N is a finite collection of nodes. Each node $n \in N$ is either associated a technology $\phi^n \in \Phi$ or extinction. At period 0 there is a unique initial node n_0: only technology ϕ^{n_0} is known, while no future risk is yet resolved so that all final nodes $N_{\bar{t}} \subset N$ can be reached from n_0. As time flows, risk resolves. At the final period $\bar{t} \in T$, the full history of technology and extinction is known. Without loss of generality, each final node $n \in N_{\bar{t}}$ is reached with positive probability $\pi^n \in (0, 1]$, with $\sum_{n \in N_{\bar{t}}} \pi^n = 1$.

At each period $t \in T$, society knows the realization of history until t. Let $N_t \subset N$ be the subset of nodes at t. Each node $n \in N_t$ is uniquely identified by the subset of final nodes $N_{\bar{t}}(n)$ that can be reached from n.[17] Extinction is irreversible. If node $n \in N$ is associated extinction, also each strict successors of n, i.e. $N(n)$, is. Let $N^\ell \subseteq N$ be the tree obtained from N by eliminating the nodes that are associated extinction. For each period $t \in \{1, ..., \bar{t}\}$, the number of non-extinction nodes is larger than 3. Let $\pi_t \in (0, 1]$ be the (unconditional) existence probability of generation t.

An **allocation** x specifies a consumption x^n for each node $n \in N_t^\ell$ and each generation $t \in T$.[18] By construction, the assignment of generation t at node n can only depend

[15] Vector inequalities are defined as follows: $x \geq y \Leftrightarrow [x_i \geq y_i \,\forall i]$; $x > y \Leftrightarrow [x \geq y$ and $x \neq y]$; and $x \gg y \Leftrightarrow [x_i > y_i \,\forall i]$.

[16] With minor changes, the results extend to models with infinite time horizon and/or exogenous population dynamics. Difficulties with infinite time horizon are well-known since at least Diamond (1965), but can be addressed (among others) by weakening "efficiency" Zuber and Asheim (2012). To tackle exogenous population dynamics, the later-introduced transfer principles need to account for the number of individuals involved in the transfers and lead to society weighing generations by their size.

[17] As standard, this requires that later partitions of possible histories are finer. Formally, for each $t \in T$, each $n \in N_t$, and each $n' \in N_{t+1}$, either $N_{\bar{t}}(n) \supseteq N_{\bar{t}}(n')$ or $N_{\bar{t}}(n) \bigcap N_{\bar{t}}(n') = \emptyset$.

[18] Importantly, results are unaffected when x^n is interpreted as some measure of well-being that gener-

on the information available at n, which consists of: *(i)* the technology realized and the consumption and investment decisions taken until n; and *(ii)* the structure, the intensity, and the time resolution of technological risk, summarized by the event tree N. The domain of allocations is $X \equiv \mathbb{R}_{++}^{|N^\ell|}$.

An allocation $x \in X$ is **feasible** if there exist a saving plan $s \equiv (s^n)_{n \in N^\ell}$ such that: *(i)* for each period $t \in T$ and each node $n \in N_t^\ell$, $\phi^n(k^n) \geq x^n + s^n$; *(ii)* for each period $t \in \{1, ..., \bar{t}\}$ and each node $n \in N_t^\ell$, $k^n = s^{n^-}$, where n^- denotes the unique predecessor of n; and *(iii)* $k^{n_0} = k_0$. Let $X^f \subset X$ be the set of feasible allocations.

The problem of society is to define a complete and transitive **social ranking** of allocations. Let R denote such ranking; then, $x \, R \, x'$ means that allocation x is socially at least as desirable as allocation x'. The strict preference relation P and the indifference relation I are the asymmetric and symmetric counterparts of R.

3.2. The identification of the reference

When ranking allocations, society is guided by two main objectives of distributive justice. The first is related to an effective use of resources. The most appealing allocation according to this objective can be defined as follows.

Efficiency: *An allocation $x \in X^f$ satisfies efficiency if there exist no allocation $x' \in X^f$* such that $x' > x$.

The second is related to the inequality in the distribution of resources. The most appealing allocation according to this objective can be defined as follows.

Equality: *An allocation $x \in X^f$ satisfies equality if, for each $n, n' \in N^\ell$, $x^n = x^{n'}$.*

Unfortunately, these two objectives are compatible only in the absence of risk. The intuition goes as follows. Let $C \equiv \left\{ c \in \mathbb{R}_+ \left| (c, ..., c) \in X^f \right. \right\}$. By the assumptions on technology, this set has a maximal element $\bar{c} \in C$. In the absence of risk, the allocation that assigns $\bar{c}$ to each generation satisfies efficiency and, by construction, also equality. The compatibility between efficiency and equality does not extend to the presence of risk. The example in Section 2 illustrates this difficulty. For the same resources saved in period 0, the amount of consumption that can be distributed at period 1 differs across states. Thus, any efficient distribution of resources cannot assign the same consumption to each generation at each node.

Giving priority to efficiency, I suggest weakening *equality*.[19] Equity is then interpreted as follows: the consumption assigned to each generation at each node should be as desirable

ation t enjoys at node n. Clearly, the interpretation of the welfare criterion and of the axioms is correspondingly changed.

[19] The characterization of the social ranking is independent of the allocation selected for the reference. When weakening *efficiency* (instead of *equality*), the egalitarian reference assigns the same consumption to each generation at each node and, by the isoelastic representation, drops from the social ranking. In this case, an isoelastic version of the EDU criterion (with disentanglement of the risk and time dimensions) emerges.

as the lottery over consumption assigned to later generations, restricted to the states of nature that can still occur. This way to identify the reference is closely related to the concept of sustainability proposed by Asheim and Brekke (2002).

Recursive equity: *An allocation $x \in X^f$ satisfies recursive equity if there exists a concave function $\mu : \mathbb{R}^+ \to \mathbb{R}$ such that for each $t, s \in T$ with $s > t$ and each $n \in N_t^\ell$, $x^n = \mu^{-1}\left(E\left[\mu\left(x_s\left(n\right)\right)\right]\right)$, where E is the expectation operator and $x_s\left(n\right)$ is the random variable that takes value $x^{\bar{n}}$ if $\bar{n} \in N_s^\ell\left(n\right)$ occurs.*

The following result states that efficiency is compatible with recursive equity. Moreover, these principles identify a unique allocation, denoted r. The proof is in the appendix.

Proposition 1. *There exists a unique allocation $r \in X^f$ that satisfies efficiency and recursive equity.*

3.3. The social ranking: Axioms

The first axiom is related to efficiency. Among two allocations, society prefers the one which assigns more consumption.

Monotonicity: *For each pair $x, \bar{x} \in X$, $x > \bar{x}$ implies that $x \, P \, \bar{x}$.*

Next, the social ranking is required to be continuous. Small changes of the allocation are associated small changes in the level of social welfare.

Continuity: *For each $x \in X$, the sets $\{\bar{x} \in X \,|\bar{x} \, R \, x\}$ and $\{\bar{x} \in X \,|x \, R \, \bar{x}\}$ are closed.*

The next two axioms are central to the present analysis of intergenerational ethics. They convey the idea that *some* inequalities, measured by contrast to the reference, are bad for society and reduce (or at least cannot increase) social welfare.

The first deals with "ex-ante inequalities." Comparing the assignments of two generations, say t and t', there is an **ex-ante inequality** if, at each state of nature, t is assigned more than at the equitable reference, while t' is assigned less than at the equitable reference. Generation t is then considered better-off than generation t'. The next axiom requires that society does not rank higher allocations with larger ex-ante inequality. The formalization is similar in spirit to Dalton (1920)'s transfer principle.

Ex-ante (intergenerational) equity: *For each pair $x, \bar{x} \in X$, each pair $t, t' \in T$, and each $\varepsilon \in \mathbb{R}_+$, if*

(i) $x^n = \bar{x}^n - \frac{\varepsilon}{\pi_t} \geq r^n$ *for each* $n \in N_t^\ell$;

(ii) $x^n = \bar{x}^n + \frac{\varepsilon}{\pi_{t'}} \leq r^n$ *for each* $n \in N_{t'}^\ell$;

(iii) $x^n = \bar{x}^n$ *for each* $n \in N^\ell \backslash \left\{N_t^\ell \bigcup N_{t'}^\ell\right\}$;

then $x \, R \, \bar{x}$.

The axiom reads as follows. At allocation $\bar{x}$, generation t is assigned a larger consumption than at the reference in each state (Condition *i*); generation t' is assigned a smaller consumption than at the reference in each state (Condition *ii*). Define a transfer ε from t to t' which is: weighted by the respective extinction-probabilities; uniform across states; and such that the ex-ante inequality is only reduced (but not overturned) by the transfer. Then, *ceteris paribus* (Condition *iii*), the after-transfer allocation x is socially at least as desirable as allocation $\bar{x}$.

The second equity axiom deals with "ex-post inequalities." Assume all generations but t are assigned the consumption corresponding to the reference. An **ex-post inequality** arises when generation t is assigned more than at the reference at one node and less than at the reference at another node. If the first node is reached, generation t is going to be better off than later generations; if the latter node is reached, generation t is going to be worse off than later generations. In either cases, some inequality occurs. The next axiom requires that society does not rank higher allocations with larger ex-post inequalities. The formalization is similar to a mean preserving spread (Rothschild and Stiglitz, 1970).[20] For each $t \in T$ and each $n \in N_t^\ell$, the unconditional probability that node n is reached is $\pi^n \equiv \sum_{\bar{n} \in N_{\bar{t}}(n)} \pi^{\bar{n}}$.

Ex-post (intergenerational) equity: *For each pair $x, \bar{x} \in X$, each $t \in T$, each pair $n, n' \in N_t^\ell$, and each $\varepsilon \in \mathbb{R}_+$, if*

(i) $x^n = \bar{x}^n - \frac{\varepsilon}{\pi^n} \geq r^n$;

(ii) $x^{n'} = \bar{x}^{n'} + \frac{\varepsilon}{\pi^{n'}} \leq r^{n'}$;

(iii) $x^{\tilde{n}} = \bar{x}^{\tilde{n}} = r^{\tilde{n}}$ *for each* $\tilde{n} \in N^\ell \setminus \{n, n'\}$;

then $x \, R \, \bar{x}$.

The axiom reads as follows. At allocation $\bar{x}$, generation t's assignment at node n is larger than at the reference (Condition *i*); generation t's assignment at node n' is instead smaller than at the reference (Condition *ii*). At all other nodes, generations are assigned the reference consumptions both at x and $\bar{x}$ (Condition *iii*). Define a transfer ε from n to n' which is: weighted by the probability that these nodes occur; and such that the ex-post inequality is only reduced (but not overturned) by the transfer. Then, the after-transfer allocation x is socially at least as desirable as the initial one $\bar{x}$.

Next, the social ranking should be invariant to scale changes in individual consumptions. This axiom ensures that society's distributional concern is limited to the ratio (and not the absolute level) of generations' assignments.[21]

[20] The mean preserving spread is obtained by transferring probability mass to the tales of the distribution, but can be equivalently expressed as a regressive transfer across states of nature, weighted by the likelihood of each. See Atkinson (1970).

[21] As clarified by Blackorby and Donaldson (1982), this axiom "involves picking an interpersonally significant norm such as a poverty line...and the positivity restriction prevents the use of this information by assuming that everyone is above..." this poverty line. In the present setting, this poverty line

Ratio-scale invariance: *For each pair $x, \bar{x} \in X$ and each $\alpha > 0$, $x \, R \, \bar{x}$ if and only if $\alpha x \, R \, \alpha \bar{x}$.*

The last two axioms introduce some type of separability in the evaluation. The first separability axiom is across time. Assume generation t's assignment is the same at two allocations x and $\bar{x}$. Then, the social ranking of these allocations should not depend on generation t's assignment. This requirement is standard in the literature and is closely related to "independence of the utility of the dead" (Blackorby et al. (2005)).

Intergenerational separability: *For each $x, \bar{x}, \tilde{x}, \hat{x} \in X$ and each $t \in T$ such that:*

(i) $x^n = \bar{x}^n$ *and* $\tilde{x}^n = \hat{x}^n$ *for each* $n \in N_t^\ell$;

(ii) $x^n = \tilde{x}^n$ *and* $\bar{x}^n = \hat{x}^n$ *for each* $n \in N^\ell \backslash N_t^\ell$;

then $x \, R \, \bar{x}$ *if and only if* $\tilde{x} \, R \, \hat{x}$.

The second separability condition is across nodes, but within a period of time. Consider two allocations x and $\bar{x}$ that assign the same consumption to each generation but to generation t. If furthermore t's assignment at a node n is unaffected by the choice, the consumption assigned at n should be irrelevant for the ethical assessment.

Intragenerational separability: *For each $x, \bar{x}, \tilde{x}, \hat{x} \in X$, each $t \in N$, and each $n \in N_t^\ell$ such that:*

(i) $x^n = \bar{x}^n$ *and* $\tilde{x}^n = \hat{x}^n$;

(ii) $x^{n'} = \tilde{x}^{n'}$ *and* $\bar{x}^{n'} = \hat{x}^{n'}$ *for each* $n' \in N_t^\ell \backslash \{n\}$;

(iii) $x^{n''} = \bar{x}^{n''} = \tilde{x}^{n''} = \hat{x}^{n''}$ *for each* $n'' \in N^\ell \backslash N_t^\ell$;

then $x \, R \, \bar{x}$ *if and only if* $\tilde{x} \, R \, \hat{x}$.

Ratio-scale invariance and the separability axioms are demanding requirements. Yet, these have valuable implications and are, thus, common in the literature. First, they provide informational parsimony: the comparison of allocations only requires information about the relative consumptions of the generations affected by the choice. Second, these significantly simplify the social welfare function and, thus, can be easier applied for the computation of optimization problems. Finally and most importantly, they ensure tractability of the representation result and help understanding the effects of the reference on the social evaluation.

corresponds to a 0 consumption and can be interpreted as the consumption associated to a life barely worth living.

3.4. The social ranking: the representation result

I first present the formulation of the criterion. For each $t \in T$, define the **(expected) reference consumption of generation t** as the average consumption that generation t enjoys at the reference conditional on existence, i.e. $r_t \equiv \sum_{n \in N_t^\ell} \pi^n r^n$. For each $t \in T$, let $\gamma_t \geq 0$ and define the power function v_t by setting, for each $z \in \mathbb{R}_+$, $v_t(z) = z^{1-\gamma_t}$ if $\gamma_t \neq 1$ and $v_t(z) = \ln z$ if $\gamma_t = 1$. Then, the **welfare of generation t** at $x \in X$ is given by:

$$W_t(x;r) \equiv v_t^{-1}\left[\frac{1}{r_t}\sum_{n \in N_t^\ell} \pi^n r^n v_t\left(\frac{x^n}{r^n}\right)\right]. \tag{8}$$

What matters for the welfare of each generation is not the absolute level of consumption, but the ratio between the assignment and the equitable reference at each node. The parameter γ_t of the power function v_t measures the **ex-post inequality aversion at t**: the higher γ_t the less society is willing to accept differences between x^n and r^n. This parameter may differ across time since each period is different in terms of the risk: in principle, society might choose a time-dependent aversion to ex-post inequality to reflect the different type and intensity of risk that characterizes each period. When v_t is linear (or $\gamma_t = 0$), the equitable reference is irrelevant and the welfare of generation t is given by the expected consumption it is assigned, i.e. $W_t(x;r) = \sum_{n \in N_t^\ell} \pi^n x^n$; at the limit for v_t being infinitely concave (or $\gamma_t \to \infty$), the welfare of generation t is measured by the lowest ratio between the assigned and the reference consumption, i.e. $W_t(x;r) = \min_{n \in N_t^\ell} \frac{x^n}{r^n}$.

Let $\rho \geq 0$ and define the power function w by setting, for each $z \in \mathbb{R}_+$, $w(z) = \dfrac{z^{1-\rho}}{1-\rho}$ if $\rho \neq 1$ and $w(z) = \ln z$ if $\rho = 1$. Then, **intergenerational social welfare** at $x \in X$ is given by:[22]

$$W(x;r) \equiv \sum_{t \in T} r_t w(W_t(x;r)). \tag{9}$$

The welfare of each generation is transformed by the concave power function w, weighted by the reference consumption, and then additively aggregated. The parameter ρ defining the function w measures the **ex-ante inequality aversion**: the higher ρ, the less society is willing to accept differences between each generation's welfare.

The innovative aspect of this specification lies in the weights r_t assigned to each generation's welfare. The reference consumption r_t captures two important aspects of each risky intergenerational problem: the likelihood that generation t exists; and the riskiness of technology at t. The first aspect constitutes a standard argument for discounting later generations. The larger the probability of extinction, the smaller the reference consumption r_t and, consequently, the smaller the weight generation t's welfare is given. The second aspect is new. The more risky the technology in a later period, the more the egali-

[22] Note that this formulation is welfare equivalent to that of eq. (1) where the weights are normalized with respect to the reference consumption of generation 0.

tarian reference balances such risk with a larger expected consumption and, consequently, the larger the weight generation t's welfare is given. The interplay of these two aspects determines social discounting. Interestingly, a later generation might be assigned a larger weight than an earlier one (corresponding to the case of negative "risk adjusted" discount rates) when the second aspect dominates the first one.

I can now define the welfare criterion.

The social ranking is ***reference-dependent utilitarian*** *if it can be numerically represented by a social welfare function W as defined in (9); that is, for each pair of allocations $x, \bar{x} \in X$:*

$$x \, R \, \bar{x} \Leftrightarrow W\left(x; r\right) \geq W\left(\bar{x}; r\right).$$

The main result establishes the equivalence between the above-introduced axioms and the *reference-dependent utilitarian* criterion.[23] The proof is in the appendix.

Theorem 1. *A social ranking satisfies monotonicity, continuity, ex-ante equity, ex-post equity, ratio-scale invariance, intergenerational separability, and intragenerational separability if and only if it is reference-dependent utilitarian.*

4. Conclusions

In the literature, welfare issues involving intergenerational risks are generally addressed by analogy to Harsanyi (1955)'s pioneering contribution to the evaluation of risky social situations. Agents are simply reinterpreted as generations and time discounting is added. I claim that such an approach disregards essential aspects of intergenerational risks. First, risk resolves gradually over time. Second, it exposes generations to different types and quantity of risk. Third, risk is, to a large extent, uninsurable. Consequently, it naturally generates inequalities across generations, independent of the state of nature that eventually occurs.

The principles of justice introduced here take into account these aspects of intergenerational risk. The first step is to choose a reference allocation. This allocation is identified as the most equitable among the efficient allocations. It thus accounts for the time resolution of risk, the heterogeneous risk faced by the generations, and the unavoidable inequalities among generations. The second step is to assess allocations by contrast to such reference. More specifically, the main principles introduced here tell that more inequalities—as measured with respect to the reference—cannot improve social welfare.

The axiomatic analysis singles out the class of *reference-dependent utilitarian* criteria.

[23] This result is independent of the choice of the reference. More precisely, the *reference-dependent utilitarian* criterion does not rely on the reference beeing chosen based on *efficiency* and *recursive equity*. For instance, combining *equality* with a weakening of *efficiency*, the reference would assign the same consumption at each node. The corresponding *reference-dependent utilitarian* criterion would then simplify to an isoelastic additive criterion with different elasticities of substitution over time and states.

These criteria avoid serious drawbacks of EDU, related to: *(i)* the choice of the correct rate of social discounting; and *(ii)* the capacity of the criterion to accommodate social concern to distributional issues. The drawback of *reference-dependent utilitarianism* is its time inconsistency; this issue seems, however, not fatal and only requires society to be sophisticated in its policy choices (Pollak, 1968; Asheim and Mitra, 2016).

Several important features of intergenerational problems remain unaddressed and require further investigation. The analysis does not consider endogenous population issues (see Blackorby et al. (2005)), which might substantially aggravate future resource scarcity. Moreover, the "event tree" structure of information disclosure does not allow society to address unawareness about future events (see Dekel et al. (1998)). Finally, the restriction to a single dimension of well-being, with neither overlapping generations nor multiple commodities, rules out the ethical difficulties of confronting conflicting views about what constitutes a good life (see Piacquadio (2014)) and might lead to underestimating the effects of environmental damages (see Sterner and Persson (2008)).

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A. Proofs

A.1. Proposition 1

Proof. I first show existence. Let $X^{f+} \subset \mathbb{R}_+^{|N^\ell|}$ be such that $x \in X^{f+}$ if there exists $x' \in X^f$ with $x' \geq x$.

Define $X^{RE} \subseteq X^{f+}$ as the subset of allocations satisfying *recursive equity.* Let $C_0 \equiv \{c \in \mathbb{R}_+ \,|x_0 = c \text{ for some } x \in X^{RE}\}$. The set C_0 is non-empty: by assumption $X^f \neq \emptyset$ and, since the production functions are strictly increasing, concave, and satisfy no free lunch, there exists $x \in X^f$ and $c > 0$ such that, for each $n \in N^\ell$, $x^n = c$; thus $x_0 = c \in C_0$. The set C_0 is bounded: this immediately follows from X^f and X^{f+} being bounded. The set C_0 is compact: this follows from the continuity of technology F, the concavity of function μ, and the compactness of X^{f+}. Let $x^* \in X^{RE}$ be such that x_0^* is the maximal element of C_0. By construction, x^* satisfies *recursive equity.* By contradiction, assume that x^* does not satisfy *efficiency*: then there exists $x' \in X^f$ such that $x' > x$. Then, by the mentioned assumptions on technology, there exists a $x'' \in X^{RE}$ such that $x'' \gg x^*$, contradicting x_0^* being a maximal element of C_0. Finally, since $x_0^* > 0$ and technology is strictly increasing and continuous, $x^* \gg 0$ and, thus, $x^* \in X^f$.

I next show uniqueness. By contradiction, assume there exists a pair $x, \bar{x} \in X^f$ with $x \neq \bar{x}$ that satisfy *efficiency* and *recursive equity.* Let $t \in T$ be the first period for which $x^n \neq \bar{x}^n$ for some $n \in N_t^\ell$. If $t = 0$, $x_0 \gtrless \bar{x}_0$ and the same argument as above leads to a contradiction of *efficiency* for one of the two allocations. Assume $t > 0$ and define:

$$X^{RE}(n) \equiv \left\{ \hat{x} \in X^{RE} \middle| \hat{x}^{n'} = x^{n'} = \bar{x}^{n'} \text{ for each } n' \in N_{t'}^\ell \text{ with } t' < t \right\}, \text{ and}$$

$$C(n) \equiv \left\{ c \in \mathbb{R}_+ \middle| c = \hat{x}^n \text{ for some } \hat{x} \in X^{RE}(n) \right\}.$$

Again, the same reasoning as for C_0 allows concluding that either $x^n = \bar{x}^n$ or one of the two allocation does not satisfy *efficiency.* A contradiction. □

A.2. Theorem 1

Part 1. *If a social ranking satisfies the axioms, then it is reference-dependent utilitarian.*

The argument is divided in several steps. The proof of each can be found in the online appendix unless stated otherwise. Assume the social ranking satisfies *monotonicity, continuity, ex-ante equity, ex-post equity, ratio-scale invariance, intergenerational separability,* and *intragenerational separability.*

The first step shows that the social ranking R admits a specific functional representation, which is continuous, increasing, and additive over time and, for each period, additive across nodes. This is an implication of *monotonicity, continuity, intergenerational separability,* and *intragenerational separability.*

Step 1. *For each $t \in T$ and each $n \in N_t^\ell$, there exist continuous and strictly increasing functions q_t and $\bar{v}^n$ such that R is represented by:*

$$V(x; r) = \sum_{t \in T} q_t \left(\sum_{n \in N_t^\ell} \bar{v}^n(x^n) \right). \tag{10}$$

The next step highlights that, by *ex-post equity* and *ratio-scale invariance*, the function $\bar{v}^n$ is a concave transformation of the "relative consumption" x^n/r^n and is equal across nodes belonging to the same period (up to an additive constant).

Step 2. *For each $t \in T$, there exist strictly increasing and concave function $v_t : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $x^n \in \mathbb{R}_+$, each $n \in N_t^\ell$, and some $\chi^n \in \mathbb{R}$:*

$$\bar{v}^n(x^n) = \pi^n r^n v_t\left(\frac{x^n}{r^n}\right) + \chi^n.$$

By imposing *ratio-scale invariance*, the next step proves that v_t is a power function.

Step 3. *For each $t \in T$, there exist constants $\eta_t \in \mathbb{R}_+$ and $\gamma_t, \bar{\eta}_t \in \mathbb{R}$ such that for each $a \in \mathbb{R}_+$:*

$$v_t(a) = \frac{\eta_t}{1-\gamma_t} a^{1-\gamma_t} + \bar{\eta}_t \text{ if } \gamma_t \neq 1 \text{ and}$$
$$v_t(a) = \eta_t \ln a + \bar{\eta}_t \text{ if } \gamma_t = 1.$$

For each $t \in T$, let $q_t(x;r) \equiv q_t\left(\sum_{n \in N_t^\ell} \bar{v}^n(x^n)\right)$. Again by *ratio-scale invariance*, $q_t(x;r)$ can be written as a product of a function ψ_t (to be identified in the subsequent step) and a specific positively linearly homogeneous function of x.

Step 4. *For each $t \in T$, there exists an increasing functions $\psi_t : \mathbb{R} \to \mathbb{R}$ such that, for each $x \in X$,*

$$q_t(x;r) = \psi_t\left[(1-\gamma_t)\left(\frac{\eta_t}{1-\gamma_t}\sum_{n \in N_t^\ell} \pi^n r^n \left(\frac{x^n}{r^n}\right)^{1-\gamma_t}\right)^{\frac{1}{1-\gamma_t}}\right] \text{ if } \gamma_t \neq 1 \text{ and}$$

$$q_t(x;r) = \psi_t\left[\exp\left(\eta_t \sum_{n \in N_t^\ell} \pi^n r^n \ln\left(\frac{x^n}{r^n}\right)\right).\right] \text{ if } \gamma_t = 1.$$

Next, by *ratio-scale invariance*, also the function ψ_t needs to have a power form.

Step 5. *There exists $\rho \in \mathbb{R}$ and, for each $t \in T$, $\xi_t \in \mathbb{R}_+$, such that for each $a \in \mathbb{R}$ and each $t \in T$:*

$$\psi_t(a) = \frac{\xi_t}{1-\rho} a^{1-\rho} \text{ if } \rho \neq 1,$$
$$\psi_t(a) = \xi_t \ln a \text{ if } \rho = 1.$$

Next, I impose *ex-ante equity* to determine restrictions on ρ and the parameters ξ_t and η_t.

Step 6. The following parameter restrictions hold: $\rho \geq 0$ and, for each $t \in T$, $\xi_t = \eta_t^{-1}$ and $\eta_t = \frac{1-\gamma_t}{r_t}$ if $\gamma_t \neq 1$ and $\eta_t = \frac{1}{r_t}$ otherwise.

The last step combines the previous ones.

Step 7. *Steps 1-6 imply that the social ranking is reference-dependent utilitarian.*

Introducing the determined restrictions on parameters and substituting the functional forms obtained in Steps 2-6 into the additive representation from Step 1 directly proves the result.

Part 2. *The reference-dependent utilitarian criterion satisfies the axioms.*

Since the welfare criterion is increasing in the assigned utilities, it satisfies *monotonicity*. Since it is continuous, it satisfies *continuity*. Since it is homogeneous with respect to the allocation, it satisfies *ratio-scale invariance*. Since it is additive over each generation's welfare, it satisfies *intergenerational separability*. Since for each $t \in T$, the assigned consumptions enter additively in $W_t(x; r)$, the welfare criterion also satisfies *intragenerational separability*. The implications for *ex-post equity* and *ex-ante* are presented as lemmas. The proofs can be found in the online appendix.

Lemma 1. *If a social ordering is reference-dependent utilitarian, then it satisfies ex-post equity.*

Lemma 2. *If a social ranking is reference-dependent utilitarian, then it satisfies ex-ante equity.*

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B. FOR ONLINE PUBLICATION: The stochastic social discount rate

This section derives the stochastic social discount rate (see also Traeger (2014)). The project trades a deterministic unit of consumption in period 0, i.e. dx_0, against a fraction $d\varepsilon$ of a stochastic project A, defined by the technology. Formally, the stochastic social discount rate is characterized as the value

$$d \equiv \ln \frac{d\varepsilon}{-dx_0}, \tag{11}$$

that leaves overall welfare constant. For the *reference-dependent utilitarian* criterion, the stochastic social discount rate $d(x;r)$ depends on both the allocation x and the reference r. To ensure that social welfare is unchanged, $d\varepsilon$ and dx_0 satisfy:

$$\frac{d}{dx_0} w\left(\frac{x_0}{r_0}\right) dx_0 + \frac{E[r_1]}{r_0}\pi \cdot \frac{d}{d\varepsilon} w \circ v^{-1} \left(\frac{E\left[r_1 v\left(\frac{x_1+\varepsilon A}{r_1}\right)\right]}{E[r_1]} \right)\Bigg|_{\varepsilon=0} d\varepsilon = 0. \tag{12}$$

Define

$$W_1(\varepsilon) \equiv w \circ v^{-1} \left(\frac{E\left[r_1 v\left(\frac{x_1+\varepsilon A}{r_1}\right)\right]}{E[r_1]} \right) = \frac{1}{1-\rho} \left(\frac{E\left[r_1^{\gamma}(x_1+\varepsilon A)^{1-\gamma}\right]}{E[r_1]} \right)^{\frac{1-\rho}{1-\gamma}}.$$

Then, (12) can be written as:

$$\frac{d}{dx_0} w\left(\frac{x_0}{r_0}\right) dx_0 + \frac{E[r_1]}{r_0}\pi \cdot \frac{d}{d\varepsilon} \left. W_1(\varepsilon)\right|_{\varepsilon=0} d\varepsilon = 0. \tag{13}$$

The effect of a marginal change in consumption at 0, using the definition of w, is:

$$\frac{d}{dx_0} w\left(\frac{x_0}{r_0}\right) = r_0^{-1+\rho} x_0^{-\rho}. \tag{14}$$

The effect of a marginal stochastic project $d\varepsilon$ is:

$$\frac{d}{d\varepsilon} \left. W_1(\varepsilon)\right|_{\varepsilon=0} = \left(\frac{E\left[r_1^{\gamma} x_1^{1-\gamma}\right]}{E[r_1]} \right)^{\frac{1-\rho}{1-\gamma}-1} \frac{E\left[A r_1^{\gamma} x_1^{-\gamma}\right]}{E[r_1]}. \tag{15}$$

B.1. Stochastic social discount rate: a simple case

Assume that x is a non-wasteful distribution of resources. Then, $x_1 = A(\omega - x_0)$ and, by *efficiency* of the reference, also $r_1 = A(\omega - r_0)$. Then, (15) significantly simplifies as:

$$\begin{aligned} \frac{d}{d\varepsilon} W_1(\varepsilon)\Big|_{\varepsilon=0} &= \left(\frac{E\left[A(\omega-r_0)^{\gamma}(\omega-x_0)^{1-\gamma}\right]}{E[A(\omega-r_0)]}\right)^{\frac{1-\rho}{1-\gamma}-1} \frac{E\left[A(\omega-r_0)^{\gamma}(\omega-x_0)^{-\gamma}\right]}{E[A(\omega-r_0)]} \\ &= \left(E[r_1]^{-(1-\gamma)} E[x_1]^{1-\gamma}\right)^{\frac{1-\rho}{1-\gamma}-1} E[A]\, E[r_1]^{-(1-\gamma)} E[x_1]^{-\gamma} \\ &= E[r_1]^{\rho-1} E[x_1]^{-\rho} E[A]. \end{aligned}$$

Substituting the above expression and (14) in (13), gives:

$$r_0^{-1+\rho} x_0^{-\rho} dx_0 + \frac{E[r_1]}{r_0} \pi E[r_1]^{\rho-1} E[x_1]^{-\rho} E[A]\, d\varepsilon = 0,$$

and, simplifying,

$$\frac{d\varepsilon}{-dx_0} = \pi^{-1} E[A]^{-1} \left[\frac{E[r_1]}{r_0}\right]^{\rho} \left[\frac{E[x_1]}{x_0}\right]^{-\rho}.$$

Substuting in (11) and, considering that $\bar{g}_r \equiv \ln\left[\frac{E[r_1]}{r_0}\right]$ and that $\bar{g}_x \equiv \ln\left[\frac{E[x_1]}{x_0}\right]$, the social discount rate becomes:

$$d(x; r) = \ln \frac{d\varepsilon}{-dx_0} = -\ln\pi - \ln E[A] + \rho(\bar{g}_x - \bar{g}_r).$$

Defining $\delta \equiv -\ln\pi$ and $\alpha \equiv -\ln E[A]$, gives the social discount rate (7). Note that, in this case, no assumption on the distribution of A is needed.

B.2. Social discount rate: the general case

From now on, the return on the investment A and the growth rate at x are assumed to be jointly log-normally distributed. Since $r_1 = A(\omega - r_0)$, this is equivalent to demanding that the growth rate at r, i.e. $g_r \equiv \ln\left(\frac{r_1}{r_0}\right)$, and the growth rate at x, i.e. $g_x \equiv \ln\left(\frac{x_1}{x_0}\right)$, are jointly log-normally distributed with correlation c and standard deviations σ_r and σ_x.

Now, using $r_1 = A(\omega - r_0)$, rewrite (15) as:

$$\frac{d}{d\varepsilon} W_1(\varepsilon)\Big|_{\varepsilon=0} = \left(\frac{E\left[r_1^{\gamma} x_1^{1-\gamma}\right]}{E[r_1]}\right)^{\frac{1-\rho}{1-\gamma}-1} \frac{E\left[r_1^{1+\gamma} x_1^{-\gamma}\right]}{E[r_1](\omega - r_0)}. \tag{16}$$

By the definition of the growth rates and their distributional assumptions,

$$\begin{aligned} E\left[r_1^{\gamma} x_1^{1-\gamma}\right] &= r_0^{\gamma} x_0^{1-\gamma} E\left[e^{\gamma g_r + (1-\gamma) g_x}\right] \\ &= r_0^{\gamma} x_0^{1-\gamma} e^{\gamma \bar{g}_r + (1-\gamma)\bar{g}_x + \gamma^2 \frac{\sigma_r^2}{2} + (1-\gamma)^2 \frac{\sigma_x^2}{2} + \gamma(1-\gamma) c \sigma_r \sigma_x}. \end{aligned}$$

Similarly,

$$\begin{aligned} E\left[r_1^{1+\gamma}x_1^{-\gamma}\right] &= r_0^{1+\gamma}x_0^{-\gamma}E\left[e^{(1+\gamma)g_r-\gamma g_x}\right] \\ &= r_0^{1+\gamma}x_0^{-\gamma}e^{(1+\gamma)\bar{g}_r-\gamma\bar{g}_x+(1+\gamma)^2\frac{\sigma_r^2}{2}+\gamma^2\frac{\sigma_x^2}{2}-\gamma(1+\gamma)c\sigma_r\sigma_x}. \end{aligned}$$

Thus, substituting in (16) and rearranging, yields:

$$\begin{aligned} \tfrac{d}{d\varepsilon}\,W_1(\varepsilon)\big|_{\varepsilon=0} &= (\omega-r_0)^{-1}\left(\tfrac{E[r_1]}{r_0}\right)^{-\frac{1-\rho}{1-\gamma}}\left(\tfrac{r_0}{x_0}\right)^{\rho}e^{\left(\gamma\frac{1-\rho}{1-\gamma}+1\right)\bar{g}_r-\rho\bar{g}_x} \\ & \quad e^{\left(\gamma^2\frac{1-\rho}{1-\gamma}+1+2\gamma\right)\frac{\sigma_r^2}{2}+(\gamma-(1-\gamma)\rho)\frac{\sigma_x^2}{2}-(1+\rho)\gamma c\sigma_r\sigma_x}. \end{aligned}$$

and, since $\frac{E[r_1]}{E[A]}=(\omega-r_0)^{-1}$ and $\frac{E[r_1]}{r_0}=E\left[e^{g_r}\right]=e^{\bar{g}_r+\frac{\sigma_r^2}{2}}$:

$$\tfrac{d}{d\varepsilon}\,W_1(\varepsilon)\big|_{\varepsilon=0} = \tfrac{E[A]}{E[r_1]}\left(\tfrac{r_0}{x_0}\right)^{\rho}e^{-\rho(\bar{g}_x-\bar{g}_r)-\rho\left(\frac{\sigma_x^2}{2}-\frac{\sigma_r^2}{2}\right)+\gamma(1+\rho)\left(\frac{\sigma_x^2}{2}+\frac{\sigma_r^2}{2}-c\sigma_r\sigma_x\right)}.$$

Substituting this expression in (13) and solving for $\frac{d\varepsilon}{-dx_0}$, leads to the following social discount rate:

$$d(x;r)=\ln\frac{d\varepsilon}{-dx_0}=\delta+\alpha+\rho(\bar{g}_x-\bar{g}_r)+\rho\left(\frac{\sigma_x^2}{2}-\frac{\sigma_r^2}{2}\right)-\gamma(1+\rho)\left(\frac{\sigma_x^2}{2}+\frac{\sigma_r^2}{2}-c\sigma_r\sigma_x\right),$$

where $\delta\equiv-\ln\pi$ and $\alpha\equiv-\ln E[A]$.

B.3. Stochastic social discount rate: the EDU criterion

For the EDU criterion, the social discount rate is the value

$$\bar{d}(x)\equiv\ln\frac{d\varepsilon}{-dx_0}, \tag{17}$$

for which $d\varepsilon$ and dx_0 satisfy:

$$\frac{d}{dx_0}\frac{x_0^{1-\eta}}{1-\eta}dx_0+\beta\frac{d}{d\varepsilon}\;E\left[\frac{(x_1+\varepsilon A)^{1-\eta}}{1-\eta}\right]\Bigg|_{\varepsilon=0}d\varepsilon=0. \tag{18}$$

The first derivative gives:

$$\frac{d}{dx_0}\frac{x_0^{1-\eta}}{1-\eta}=x_0^{-\eta}.$$

The second derivative can be written as:

$$\frac{d}{d\varepsilon}\;E\left[\frac{(x_1+\varepsilon A)^{1-\eta}}{1-\eta}\right]\Bigg|_{\varepsilon=0}=E\left[Ax_1^{-\eta}\right].$$

Since the distributional assumption is made in terms of the growth rates at r and at x, substitute $A = r_1 (\omega - r_0)^{-1}$ and solve:

$$\begin{aligned} E\left[Ax_1^{-\eta}\right] &= \tfrac{E\left[r_1 x_1^{-\eta}\right]}{\omega - r_0} = \tfrac{r_0 x_0^{-\eta}}{\omega - r_0} E\left[e^{g_r - \eta g_x}\right] \\ &= \tfrac{r_0 x_0^{-\eta}}{\omega - r_0} e^{\bar{g}_r - \eta \bar{g}_x + \frac{\sigma_r^2}{2} + \eta^2 \frac{\sigma_x^2}{2} - \eta c \sigma_r \sigma_x}, \end{aligned}$$

and, since $\frac{E[r_1]}{E[A]} = (\omega - r_0)^{-1}$ and $\frac{E[r_1]}{r_0} = E\left[e^{g_r}\right] = e^{\bar{g}_r + \frac{\sigma_r^2}{2}}$:

$$E\left[Ax_1^{-\eta}\right] = E[A]\, x_0^{-\eta} e^{-\eta \bar{g}_x + \eta^2 \frac{\sigma_x^2}{2} - \eta c \sigma_r \sigma_x}.$$

Substituting in (18) and rearranging, yields:

$$\frac{d\varepsilon}{-dx_0} = \beta^{-1} E[A]^{-1} e^{\eta \bar{g}_x - \eta^2 \frac{\sigma_x^2}{2} + \eta c \sigma_r \sigma_x} d\varepsilon.$$

This leads the stochastic social discount rate:

$$\bar{d}(x) = \bar{\delta} + \alpha + \eta \bar{g}_x - \eta^2 \frac{\sigma_x^2}{2} + \eta c \sigma_r \sigma_x,$$

where $\bar{\delta} \equiv -\ln \beta$ and $\alpha \equiv -\ln E[A]$. This expression is equivalent to that presented in Subsection 2.4.

C. FOR ONLINE PUBLICATION: Detailed proofs

C.1. Proof of Step 1

Proof. By assumption, there are at least 3 periods and, for each period, there are at least 3 nodes of non-extinction. Since *monotonicity* implies the axiom of "strict essentiality," Gorman (1968)'s theorem on overlapping separable sets applies: "strict essentiality," *continuity, intergenerational separability*, and *intragenerational separability* imply that there exist continuous functions q_t (one for each $t \in T$) and $\bar{v}^n$ (one for each $n \in N^\ell$) such that R is represented by (10).[24] By *monotonicity*, it must be true that, for each $t \in T$ and each $n \in N_t^\ell$, either q_t and $\bar{v}^n$ are all strictly increasing or these are all strictly decreasing. Either choices lead to ordinally equivalent representations of R. □

C.2. Proof of Step 2

Proof. For each $t \in T$, each $n \in N_t^\ell$, and each $x^n \in \mathbb{R}_+$ define:

$$v^n\left(\frac{x^n}{r^n}\right) \equiv \frac{\bar{v}^n(x^n)}{\pi^n r^n}.$$

[24] "Strict essentiality" states that each individual's assignment matters for the social ranking; see also Blackorby and Donaldson (1982).

Since $\bar{v}^n$ is strictly increasing (by Step *1*), also v^n is. Let a pair $x, \bar{x} \in X$ be such that for some $t \in T$, a pair $n, n' \in N_t^\ell$, and a $\varepsilon \in \mathbb{R}_+$ the following conditions hold: *(i)* $x^n = \bar{x}^n - \frac{\varepsilon}{\pi^n} \geq r^n$; *(ii)* $x^{n'} = \bar{x}^{n'} + \frac{\varepsilon}{\pi^{n'}} \leq r^{n'}$; and *(iii)* $x^{\tilde{n}} = \bar{x}^{\tilde{n}}$ for each $\tilde{n} \in N^\ell \backslash \{n, n'\}$. By *ex-post equity*, $x \, R \, \bar{x}$. By Step *1*, this implies that $V(x; r) - V(\bar{x}; r) \geq 0$ or, using *(iii)*, that:

$$\begin{aligned} \bar{v}^n(x^n) - \bar{v}^n\left(x^n + \frac{\varepsilon}{\pi^n}\right) + \\ \bar{v}^{n'}\left(x^{n'}\right) - \bar{v}^{n'}\left(x^{n'} - \frac{\varepsilon}{\pi^{n'}}\right) \geq 0 \end{aligned} \tag{19}$$

Substituting the functions v^n and $v^{n'}$ in (19), gives:

$$\begin{aligned} \pi^n r^n \left[v^n\left(\frac{x^n}{r^n}\right) - v^n\left(\frac{x^n}{r^n} + \frac{\varepsilon}{\pi^n r^n}\right)\right] + \\ \pi^{n'} r^{n'} \left[v^{n'}\left(\frac{x^{n'}}{r^{n'}}\right) - v^{n'}\left(\frac{x^{n'}}{r^{n'}} - \frac{\varepsilon}{\pi^{n'} r^{n'}}\right)\right] \geq 0. \end{aligned}$$

If v^n and $v^{n'}$ are differentiable at $\left(\frac{x^n}{r^n}\right)$ and $\left(\frac{x^{n'}}{r^{n'}}\right)$ respectively, dividing by ε and taking the limit for $\varepsilon \to 0$, yields:

$$\left.\frac{\partial v^n(a)}{\partial a}\right|_{a=\frac{x^n}{r^n}} \leq \left.\frac{\partial v^{n'}(a)}{\partial a}\right|_{a=\frac{x^{n'}}{r^{n'}}}. \tag{20}$$

Since v^n and $v^{n'}$ are strictly increasing, these are differentiable almost everywhere. Thus, (20) holds for almost all $\left(\frac{x^n}{r^n}\right) \geq 1 \geq \left(\frac{x^{n'}}{r^{n'}}\right)$ and, symmetrically, the reverse inequality holds for almost all $\left(\frac{x^n}{r^n}\right) \leq 1 \leq \left(\frac{x^{n'}}{r^{n'}}\right)$. Thus, if the functions are differentiable at 1, $\left.\frac{\partial v^n(a)}{\partial a}\right|_{a=1} = \left.\frac{\partial v^{n'}(a)}{\partial a}\right|_{a=1}$.

By *proportionality* and Step *1*, $V(x; r) \geq V(\bar{x}; r)$ if and only if $V(bx; r) \geq V(b\bar{x}; r)$ for each $b > 0$. Thus, equation (20) holds almost everywhere for each $\left(\frac{x^n}{r^n}\right) \geq b \geq \left(\frac{x^{n'}}{r^{n'}}\right)$ and each $b > 0$. Moreover, $\left.\frac{\partial v^n(a)}{\partial a}\right|_{a=b} = \left.\frac{\partial v^{n'}(a)}{\partial a}\right|_{a=b}$ almost everywhere for each $b > 0$. This implies that for each $t \in T$, there exists a strictly increasing and concave function $v_t : \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $x^n > 0$, each $n \in N_t^\ell$, and some constant $\chi^n \in \mathbb{R}$, $v_t(x^n) = v^n(x^n) - \frac{\chi^n}{\pi^n r^n}$. Substituting the definition of v^n yields the result. □

C.3. Proof of Step 3

Proof. Let $t \in T$. Let a pair $x, \bar{x} \in X$ be such that $x^n = \bar{x}^n$ for each $n \in N^\ell \backslash N_t^\ell$. By *ratio-scale invariance*, for each $\alpha > 0$, $x \, R \, \bar{x}$ if and only if $\alpha x \, R \, \alpha \bar{x}$. By Step 1 and 2, *ratio-scale invariance* implies that

$$\sum_{n \in N_t^\ell} \pi^n r^n \left[v_t \left(\frac{x^n}{r^n} \right) - v_t \left(\frac{\bar{x}^n}{r^n} \right) \right] \geq 0$$

if and only if

$$\sum_{n \in N_t^\ell} \pi^n r^n \left[v_t \left(\frac{\alpha x^n}{r^n} \right) - v_t \left(\frac{\alpha \bar{x}^n}{r^n} \right) \right] \geq 0.$$

Since v_t is the same for each $n \in N_t^\ell$, Theorem 6 of Roberts (1980) applies: this directly implies that v_t is an increasing affine transformation of a power function. □

C.4. Proof of Step 4

Proof. Let $t \in T$. By Step 2,

$$q_t(x; r) = q_t \left(\sum_{n \in N_t^\ell} \pi^n r^n v_t \left(\frac{x^n}{r^n} \right) + \chi_t \right),$$

where $\chi_t \equiv \sum_{n \in N_t^\ell} \chi^n$. Let $x, \bar{x} \in X$ be such that $x^n = \bar{x}^n$ for each $n \in N^\ell \backslash N_t^\ell$. By *ratio-scale invariance*, for each $\alpha > 0$, $V(x; r) \geq V(\bar{x}; r)$ if and only if $V(\alpha x; r) \geq V(\alpha \bar{x}; r)$. Since V is additive over time (Step 1), this is equivalent to $q_t(x; r) \geq q_t(\bar{x}; r)$ if and only if $q_t(\alpha x; r) \geq q_t(\alpha \bar{x}; r)$. Thus q_t is homothetic with respect to x. It follows that it can be written as $q_t(x; r) = \psi_t(\tilde{q}_t(x; r))$ where ψ_t is an increasing function and $\tilde{q}_t$ is positively linearly homogeneous and such that:

$$\tilde{q}_t(x; r) = \overset{*}{q}_t \left(\sum_{n \in N_t^\ell} \pi^n r^n v_t \left(\frac{x^n}{r^n} \right) + \chi_t \right),$$

with $\overset{*}{q}_t$ continuous and increasing.

Case 1. Assume $\gamma_t \neq 1$. For each $n \in N_t^\ell$, substitute $v_t(a) = \frac{\eta_t}{1-\gamma_t} a^{1-\gamma_t} + \bar{\eta}_t$ (obtained in Step 3):

$$\tilde{q}_t(x; r) = \overset{*}{q}_t \left(\frac{\eta_t}{1-\gamma_t} \sum_{n \in N_t^\ell} \pi^n r^n \left(\frac{x^n}{r^n} \right)^{1-\gamma_t} + r_t \bar{\eta}_t + \chi_t \right).$$

Since $\tilde{q}_t(x; r)$ is positively linearly homogeneous, $\tilde{q}_t(\alpha x; r) = \alpha \tilde{q}_t(x; r)$ for each $\alpha > 0$. Thus:

$$\overset{*}{q}_t \left(\tfrac{\eta_t}{1-\gamma_t} \alpha^{1-\gamma_t} \textstyle\sum_{n \in N_t^\ell} \pi^n r^n \left(\frac{x^n}{r^n} \right)^{1-\gamma_t} + r_t \bar{\eta}_t + \chi_t \right) = \alpha \overset{*}{q}_t \left(\tfrac{\eta_t}{1-\gamma_t} \textstyle\sum_{n \in N_t^\ell} \pi^n r^n \left(\frac{x^n}{r^n} \right)^{1-\gamma_t} + r_t \bar{\eta}_t + \chi_t \right).$$

Since this holds for each $x \in X$, it follows that, for each $y \in \mathbb{R}$:

$$\overset{*}{q}_t(y) = (1-\gamma_t)(y - r_t \bar{\eta}_t - \chi_t)^{\frac{1}{1-\gamma_t}},$$

and, substituting:

$$\tilde{q}_t(x;r) = (1-\gamma_t)\left(\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n\left(\frac{x^n}{r^n}\right)^{1-\gamma_t}\right)^{\frac{1}{1-\gamma_t}}.$$

Case 2. Assume $\gamma_t = 1$. For each $n \in N_t^\ell$, substitute $v_t(a) = \eta_t \ln a + \bar{\eta}_t$ (obtained in Step 3):

$$\tilde{q}_t(x;r) = \overset{*}{q}_t\left(\eta_t\sum_{n\in N_t^\ell}\pi^n r^n \ln\left(\frac{x^n}{r^n}\right) + r_t\bar{\eta}_t + \chi_t\right).$$

By the same reasoning as above, for each $y \in \mathbb{R}$:

$$\overset{*}{q}_t(y) = \exp\left(y - r_t\bar{\eta}_t - \chi_t\right)^{\frac{1}{1-\gamma_t}},$$

and, substituting:

$$\tilde{q}_t(x;r) = \exp\left(\eta_t\sum_{n\in N_t^\ell}\pi^n r^n \ln\left(\frac{x^n}{r^n}\right)\right).$$

□

C.5. Proof of Step 5

Proof. By *ratio-scale invariance*, for each pair $x, \bar{x} \in X$ and each $\alpha > 0$, $x\,R\,\bar{x}$ if and only if $\alpha x\,R\,\alpha\bar{x}$. By Step 1 and substituting $q_t(x;r)$ for each $t \in T$, *ratio-scale invariance* implies that

$$\sum_{t\in T} q_t(x;r) \geq 0 \text{ iff } \sum_{t\in T} q_t(\alpha x;r) \geq 0.$$

By Step 4, $q_t(x;r)$ is the product of a function ψ_t and a function $\tilde{q}_t(x;r)$ that is positively linearly homogeneous with respect to x. An immediate generalization of Theorem 6 in Roberts (1980) applies: for each $t \in T$, ψ_t is an increasing affine transformation of a power function; since the function ψ_t can be different across time, each may be assigned a different positive weight ξ_t. □

C.6. Proof of Step 6

Proof. Let a pair $x, \bar{x} \in X$ be such that for some $t, t' \in T$ and $a, b, \varepsilon \in \mathbb{R}_+$:

(i) $\dfrac{x^n}{r^n} = \dfrac{\bar{x}^n}{r^n} - \dfrac{\varepsilon}{\pi_t r^n} = a \geq 1$ for each $n \in N_t^\ell$;

(ii) $\dfrac{x^n}{r^n} = \dfrac{\bar{x}^n}{r^n} + \dfrac{\varepsilon}{\pi_{t'} r^n} = b \leq 1$for each $n \in N_{t'}^\ell$;

(iii) $x^n = \bar{x}^n$ for each $n \in N^\ell \setminus \left(N_t^\ell \bigcup N_{t'}^\ell\right)$.

By *ex-ante equity*, $x\,R\,\bar{x}$ and, by Step 1 and *(iii)*:

$$q_t(x;r) - q_t(\bar{x};r) + q_{t'}(x;r) - q_{t'}(\bar{x};r) \geq 0.$$

By Steps 4 and 5, if $\gamma_t \neq 1$, then:

$$q_t(x;r) = \frac{1-\gamma_t}{1-\rho}\xi_t\left(\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n (a)^{1-\gamma_t}\right)^{\frac{1-\rho}{1-\gamma_t}};$$

$$q_t(\bar{x};r) = \frac{1-\gamma_t}{1-\rho}\xi_t\left(\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n\left(a+\frac{\varepsilon}{\pi_t r^n}\right)^{1-\gamma_t}\right)^{\frac{1-\rho}{1-\gamma_t}}.$$

Thus, dividing $q_t(x;r) - q_t(\bar{x};r)$ by ε and taking the limit for $\varepsilon \to 0$, gives:

$$\xi_t\left((a)^{1-\gamma_t}\frac{\eta_t}{1-\gamma_t}\sum_{n\in N_t^\ell}\pi^n r^n\right)^{\frac{1-\rho}{1-\gamma_t}-1}$$

$$\lim_{\varepsilon\to 0}\frac{q_t(x;r)-q_t(\bar{x};r)}{\varepsilon} = \xi_t a^{-\rho}\left(\frac{\eta_t}{1-\gamma_t}r_t\right)^{\frac{\gamma_t-\rho}{1-\gamma_t}}\eta_t,$$

Whereas, if $\gamma_t = 1$, then:

$$q_t(x;r) = \frac{1}{1-\rho}\xi_t\left(\exp\left(\eta_t\sum_{n\in N_t^\ell}\pi^n r^n \ln(a)^{1-\gamma_t}\right)\right)^{1-\rho};$$

$$q_t(x;r) = \frac{1}{1-\rho}\xi_t\left(\exp\left(\eta_t\sum_{n\in N_t^\ell}\pi^n r^n \ln\left(a+\frac{\varepsilon}{\pi_t r^n}\right)^{1-\gamma_t}\right)\right)^{1-\rho}.$$

In this case, dividing $q_t(x;r) - q_t(\bar{x};r)$ by ε and taking the limit for $\varepsilon \to 0$, gives:

$$\lim_{\varepsilon\to 0}\frac{q_t(x;r)-q_t(\bar{x};r)}{\varepsilon} = \xi_t a^{-\rho}(\eta_t r_t)^{1-\rho}\eta_t.$$

Similarly,

$$\lim_{\varepsilon\to 0}\frac{q_{t'}(x;r)-q_{t'}(\bar{x};r)}{\varepsilon} = \begin{cases} -\xi_{t'}b^{-\rho}\left(\frac{\eta_{t'}}{1-\gamma_t}r_t\right)^{\frac{\gamma_{t'}-\rho}{1-\gamma_{t'}}}\eta_{t'} & \text{if } \gamma_{t'}\neq 1\\ -\xi_{t'}b^{-\rho}(\eta_{t'}r_{t'})^{1-\rho}\eta_{t'} & \text{if } \gamma_{t'}=1.\end{cases}$$

By *ex-ante equity*,

$$\lim_{\varepsilon\to 0}\frac{q_t(x;r)-q_t(\bar{x};r)}{\varepsilon} \leq -\lim_{\varepsilon\to 0}\frac{q_{t'}(x;r)-q_{t'}(\bar{x};r)}{\varepsilon}$$

for each $a \geq 1 \geq b$ and independently of γ_t and $\gamma_{t'}$. This requires that $\rho \geq 0$;

$$\xi_t\left(\frac{\eta_t}{1-\gamma_t}r_t\right)^{\frac{\gamma_t-\rho}{1-\gamma_t}}\eta_t = \xi_t(\eta_t r_t)^{1-\rho}\eta_t = 1, \text{ and}$$

$$\xi_{t'}\left(\frac{\eta_{t'}}{1-\gamma_t}r_t\right)^{\frac{\gamma_{t'}-\rho}{1-\gamma_{t'}}}\eta_{t'}=\xi_{t'}\left(\eta_{t'}r_{t'}\right)^{1-\rho}\eta_{t'}=1,$$

which are satisfied when $\xi_t=\eta_t^{-1}$, $\xi_{t'}=\eta_{t'}^{-1}$, and

$$\eta_t=\begin{cases}\frac{1-\gamma_t}{r_t} & \text{if } \gamma_t\neq 1\\ \frac{1}{r_t} & \text{if } \gamma_t=1\text{, and}\end{cases}$$

$$\eta_{t'}=\begin{cases}\frac{1-\gamma_{t'}}{r_{t'}} & \text{if } \gamma_{t'}\neq 1\\ \frac{1}{r_{t'}} & \text{if } \gamma_{t'}=1.\end{cases}$$

□

C.7. Proof of Lemma 1

Proof. Let a pair $x,\bar{x}\in X$ be such that, for some $t\in T$, a pair $n,n'\in N_t^{\ell}$, and $\varepsilon\in\mathbb{R}_+$, the following conditions hold: *(i)* $x^n=\bar{x}^n-\frac{\varepsilon}{\pi^n}\geq r^n$; *(ii)* $x^{n'}=\bar{x}^{n'}+\frac{\varepsilon}{n'}\leq r^{n'}$; *(iii)* $x^{\tilde{n}}=\bar{x}^{\tilde{n}}$ for each $\tilde{n}\in N^{\ell}\setminus\{n,n'\}$. I need to prove that $x\,R\,\bar{x}$.

Define $a\equiv\frac{x^n}{r^n}$, $\bar{a}\equiv\frac{\bar{x}^n}{r^n}$, $b\equiv\frac{x^{n'}}{r^{n'}}$, and $\bar{b}\equiv\frac{\bar{x}^{n'}}{r^{n'}}$. By *(i)* and *(ii)* it follows that $\bar{a}>a\geq b>\bar{b}$. Condition *(iii)* implies that:

$$W(x;r)-W(\bar{x};r)\geq 0\;\Leftrightarrow W_t(x;r)-W_t(\bar{x};r)\geq 0.$$

Case $\gamma_t\neq 1$. First, let $\zeta_t\equiv 1-\gamma_t$. By condition *(iii)*, $W_t(x;r)-W_t(\bar{x};r)\geq 0$ if only if:

$$\frac{1}{\zeta_t}\left[\pi^n r^n\left(a^{\zeta_t}-\bar{a}^{\zeta_t}\right)+\pi^{n'}r^{n'}\left(b^{\zeta_t}-\bar{b}^{\zeta_t}\right)\right]\geq 0.$$

Define $\Delta\equiv\frac{\bar{a}^{\zeta_t}-\bar{b}^{\zeta_t}}{\bar{a}-\bar{b}}$. Two subcases emerge: if $\zeta_t\in(0,1]$, then $\Delta>0$; if $\zeta_t<0$, then $\Delta<0$.

Subcase $\zeta_t\in(0,1]$. By first-order approximation:

$$a^{\zeta_t}=\left(\bar{a}-\frac{\varepsilon}{\pi^n r^n}\right)^{\zeta_t}\geq\bar{a}^{\zeta_t}-\frac{\varepsilon}{\pi^n r^n}\Delta\text{ and}$$

$$b^{\zeta_t}=\left(\bar{b}+\frac{\varepsilon}{\pi^{n'}r^{n'}}\right)^{\zeta_t}\geq\bar{b}^{\zeta_t}+\frac{\varepsilon}{\pi^{n'}r^{n'}}\Delta.$$

Premultiply the first by $\pi^n r^n$ and the second by $\pi^{n'}r^{n'}$. The sum of the resulting inequalities gives:

$$\pi^n r^n\left(a^{\zeta_t}-\bar{a}^{\zeta_t}\right)+\pi^{n'}r^{n'}\left(b^{\zeta_t}-\bar{b}^{\zeta_t}\right)\geq 0.$$

Since $\zeta_t\in(0,1]$, this proves that $W_t(x;r)-W_t(\bar{x};r)$ and $x\,R\,\bar{x}$.

Subcase $\zeta_t<0$. By first-order approximation:

$$a^{\zeta_t}=\left(\bar{a}-\frac{\varepsilon}{\pi^n r^n}\right)^{\zeta_t}\leq\bar{a}^{\zeta_t}-\frac{\varepsilon}{\pi^n r^n}\Delta\text{ and}$$

$$b^{\zeta_t} = \left(\bar{b} + \frac{\varepsilon}{\pi^{n'} r^{n'}}\right)^{\zeta_t} \leq \bar{b}^{\zeta_t} + \frac{\varepsilon}{\pi^{n'} r^{n'}} \Delta.$$

Premultiply the first by $\pi^n r^n$ and the second by $\pi^{n'} r^{n'}$. The sum of the resulting inequalities gives:

$$\pi^n r^n \left(a^{\zeta_t} - \bar{a}^{\zeta_t}\right) + \pi^{n'} r^{n'} \left(b^{\zeta_t} - \bar{b}^{\zeta_t}\right) \leq 0.$$

Since $\zeta_t < 0$, this proves that $W_t(x; r) - W_t(\bar{x}; r)$ and $x \, R \, \bar{x}$.

Case $\gamma_t = 1$**.** By condition *(iii)*, $W_t(x; r) - W_t(\bar{x}; r) \geq 0$ if only if:

$$\pi^n r^n (\ln a - \ln \bar{a}) + \pi^{n'} r^{n'} \left(\ln b - \ln \bar{b}\right) \geq 0.$$

Define $\Delta \equiv \frac{\ln \bar{a} - \ln \bar{b}}{\bar{a} - \bar{b}}$. Since $\bar{a} > \bar{b}$, $\Delta > 0$. By first order linear approximation:

$$\ln a = \ln\left(\bar{a} - \frac{\varepsilon}{\pi^n r^n}\right) \geq \ln \bar{a} - \frac{\varepsilon}{\pi^n r^n} \Delta \text{ and}$$

$$\ln b = \ln\left(\bar{b} + \frac{\varepsilon}{\pi^{n'} r^{n'}}\right) \geq \ln \bar{b} + \frac{\varepsilon}{\pi^{n'} r^{n'}} \Delta.$$

Premultiply the first by $\pi^n r^n$ and the second by $\pi^{n'} r^{n'}$. The sum of the resulting gives again the required inequality, proving that $W_t(x; r) - W_t(\bar{x}; r)$ and $x \, R \, \bar{x}$. □

C.8. Proof of Lemma 2

Proof. Let a pair $x, \bar{x} \in X$ be such that for some $t, t' \in T$, with $t' > t$, and some $a \in \mathbb{R}_+$ the following conditions hold: *(i)* $\frac{x^n}{r^n} = \frac{\bar{x}^n}{r^n} - \frac{a}{\pi_t r^n} \geq 1$ for each $n \in N_t^\ell$; *(ii)* $\frac{x^n}{r^n} = \frac{\bar{x}^n}{r^n} + \frac{a}{\pi_{t'} r^n} \leq 1$ for each $n \in N_{t'}^\ell$; *(iii)* $x^{\tilde{n}} = \bar{x}^{\tilde{n}}$ for each $\tilde{n} \in N^\ell \setminus \left(N_t^\ell \bigcup N_{t'}^\ell\right)$. I need to prove that $x \, R \, \bar{x}$.

Define $\varepsilon \equiv \frac{a}{\bar{k}}$ for $\bar{k} \in \mathbb{N}_+$. Let $\left(\{x_k\}_{k \in [1, \bar{k}]}\right) \in X^{\bar{k}}$ be such that: *(I)* $x_1 = x$ and $x_{\bar{k}} = \bar{x}$; *(II)* for each $k \in \left[1, \bar{k} - 1\right]$, $\frac{x_k^n}{r^n} = \frac{x_{k+1}^n}{r^n} - \frac{\varepsilon}{\pi_t r^n}$ for each $n \in N_t^\ell$ and $\frac{x_k^n}{r^n} = \frac{x_{k+1}^n}{r^n} + \frac{\varepsilon}{\pi_{t'} r^n}$ for each $n \in N_{t'}^\ell$; *(III)* $x_k^{\tilde{n}} = x^{\tilde{n}}$ for each $\tilde{n} \in N^\ell \setminus \left(N_t^\ell \bigcup N_{t'}^\ell\right)$ and each $k \in \left[1, \bar{k}\right]$. The proof consists of showing that at the limit for $\bar{k} \to \infty$ (and thus for $\varepsilon \to 0$), $W\left(x^k; r\right) - W\left(x^{k+1}; r\right) \geq 0$; then, by transitivity, the result follows. This is done first for $\rho \neq 1$ and then for $\rho = 1$.

Case $\rho \neq 1$**.** Define $\zeta \equiv 1 - \rho$. By condition *(III)*,

$$\begin{aligned} W\left(x^k; r\right) - W\left(x^{k+1}; r\right) = \; & \frac{1}{\zeta} r_t \left[W_t\left(x^k; r\right)^\zeta - W_t\left(x^{k+1}; r\right)^\zeta\right] + \\ & \frac{1}{\zeta} r_{t'} \left[W_{t'}\left(x^k; r\right)^\zeta - W_{t'}\left(x^{k+1}; r\right)^\zeta\right]. \end{aligned} \tag{21}$$

By condition *(II)*, x^{k+1} can be written as a function of x^k and ε. Define the following functions by setting for each $\varepsilon > 0$:

$$e_t(\varepsilon) = W_t\left(x^{k+1}; r\right),$$

$$e_{t'}(\varepsilon) = W_{t'}\left(x^{k+1}; r\right).$$

Let $e_t(0) \equiv \lim_{\varepsilon \to 0} e_t(\varepsilon) = W_t(x^k; r)$ and $e_{t'}(0) \equiv \lim_{\varepsilon \to 0} e_{t'}(\varepsilon) = W_{t'}(x^k; r)$. Thus (21) can be written as:

$$W\left(x^k; r\right) - W\left(x^{k+1}; r\right) = \frac{1}{\zeta} r_t \left[e_t(0)^\zeta - e_t(\varepsilon)^\zeta\right] + \frac{1}{\zeta} r_{t'} \left[e_{t'}(\varepsilon)^\zeta - e_{t'}(0)^\zeta\right].$$

Divide by ε, and take the limit for $\varepsilon \to 0$ (or equivalently $\bar{k} \to \infty$). As $\varepsilon \to 0$, $\frac{1}{\zeta} r_t \frac{e_t(0)^\zeta - e_t(\varepsilon)^\zeta}{\varepsilon}$ tends to:

$$\frac{1}{\zeta} r_t \left. \frac{\partial}{\partial \varepsilon} e_t(\varepsilon)^\zeta \right|_{\varepsilon=0} = r_t e_t(0)^{\zeta-1} \left. \frac{\partial e_t(\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0}, \tag{22}$$

while $\frac{1}{\zeta} r_{t'} \frac{e_{t'}(0)^\zeta - e_{t'}(\varepsilon)^\zeta}{\varepsilon}$ tends to:

$$\frac{1}{\zeta} r_{t'} \left. \frac{\partial}{\partial \varepsilon} e_{t'}(\varepsilon)^\zeta \right|_{\varepsilon=0} = r_{t'} e_{t'}(0)^{\zeta-1} \left. \frac{\partial e_{t'}(\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0}. \tag{23}$$

Computing the derivatives of e_t and $e_{t'}$, yields:

$$\left. \frac{\partial e_t(\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} = -\frac{1}{\pi_t r_t} e_t(0)^{1-\gamma_t} \sum_{n \in N_t^\ell} \pi^n \left(\frac{x_k^n}{r^n}\right)^{\gamma_t - 1}; \tag{24}$$

$$\left. \frac{\partial e_{t'}(\varepsilon)}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{1}{\pi_{t'} r_{t'}} e_{t'}(0)^{1-\gamma_{t'}} \sum_{n \in N_{t'}^\ell} \pi^n \left(\frac{x_k^n}{r^n}\right)^{\gamma_{t'} - 1}. \tag{25}$$

Substituting (24) in (22), leads to:

$$\frac{1}{\zeta} r_t \left. \frac{\partial}{\partial \varepsilon} e_t(\varepsilon)^\zeta \right|_{\varepsilon=0} = -e_t(0)^{\zeta-1} \frac{\sum_{n \in N_t^\ell} \pi^n \left(\frac{x_k^n}{r^n}\right)^{\gamma_t - 1}}{\sum_{n \in N_t^\ell} \pi^n \left(e_t(0)\right)^{\gamma_t - 1}}.$$

Since $\frac{x_k^n}{r^n} \geq 1$ for each $n \in N_t^\ell$, $e_t(0) \geq 1$. Moreover $\zeta \leq 1$. Thus, $e_t(0)^{\zeta-1} \leq 1$. For the same reasons and since $\gamma_t \leq 1$, it follows that $\sum_{n \in N_t^\ell} \pi^n \left(e_t(0)\right)^{\gamma_t - 1} \geq \sum_{n \in N_t^\ell} \pi^n \left(\frac{x_k^n}{r^n}\right)^{\gamma_t - 1}$. Together, these imply that

$$\frac{1}{\zeta} r_t \left. \frac{\partial}{\partial \varepsilon} e_t(\varepsilon)^\zeta \right|_{\varepsilon=0} \geq -1.$$

Similarly, substitute (25) in (23) to get:

$$\frac{1}{\zeta} r_{t'} \left. \frac{\partial}{\partial \varepsilon} e_{t'}(\varepsilon)^\zeta \right|_{\varepsilon=0} = e_{t'}(0)^{\zeta-1} \frac{\sum_{n \in N_{t'}^\ell} \pi^n \left(\frac{x_k^n}{r^n}\right)^{\gamma_{t'} - 1}}{\sum_{n \in N_{t'}^\ell} \pi^n \left(e_{t'}(0)\right)^{\gamma_{t'} - 1}}.$$

As above (but with opposite signs), since $\frac{x_k^n}{r^n} \leq 1$ for each $n \in N_{t'}^\ell$, $e_{t'}(0) \leq 1$; moreover $\zeta \leq 1$; thus, $e_{t'}(0)^{\zeta-1} \geq 1$. For the same reasons and since $\gamma_{t'} \leq 1$, it follows that $\sum_{n \in N_{t'}^{\ell}} \pi^n \left(e_{t'}(0)\right)^{\gamma_{t'}-1} \leq \sum_{n \in N_{t'}^{\ell}} \pi^n \left(\frac{x_k^n}{r^n}\right)^{\gamma_{t'}-1}$. Together, these imply that $\frac{1}{\zeta} r_{t'} \left.\frac{\partial}{\partial \varepsilon} e_{t'}(\varepsilon)^{\zeta}\right|_{\varepsilon=0} \geq 1$.

Substituting in (21), this shows that, when $\bar{k} \to \infty$:

$$\lim_{\varepsilon \to 0} \frac{W(x_k; r) - W(x_{k+1}; r)}{\varepsilon} =$$
$$\frac{1}{\zeta} r_t \left.\frac{\partial}{\partial \varepsilon} e_t(\varepsilon)^{\zeta}\right|_{\varepsilon=0} + \frac{1}{\zeta} r_{t'} \left.\frac{\partial}{\partial \varepsilon} e_{t'}(\varepsilon)^{\zeta}\right|_{\varepsilon=0} \geq 0.$$

Since this inequality is true for each $k \in [1, \bar{k}]$, transitivity implies that $W(x_1; r) \geq W(x_{\bar{k}}; r)$ or, equivalently, $W(x; r) \geq W(\bar{x}; r)$ and $x \, R \, \bar{x}$.

Case $\rho = 1$**.** Similar steps lead to:

$$r_t \left.\frac{\partial}{\partial \varepsilon} \ln e_t(\varepsilon)\right|_{\varepsilon=0} = -e_t(0)^{-1} \frac{\sum_{n \in N_t^{\ell}} \pi^n \left(\frac{x_k^n}{r^n}\right)^{\gamma_t - 1}}{\sum_{n \in N_t^{\ell}} \pi^n \left(e_t(0)\right)^{\gamma_t - 1}} \geq -1,$$

$$r_{t'} \left.\frac{\partial}{\partial \varepsilon} \ln e_{t'}(\varepsilon)\right|_{\varepsilon=0} = e_{t'}(0)^{-1} \frac{\sum_{n \in N_{t'}^{\ell}} \pi^n \left(\frac{x_k^n}{r^n}\right)^{\gamma_{t'} - 1}}{\sum_{n \in N_{t'}^{\ell}} \pi^n \left(e_{t'}(0)\right)^{\gamma_{t'} - 1}} \geq 1.$$

Thus, when $\bar{k} \to \infty$, $\lim_{\varepsilon \to 0} \frac{W(x_k; r) - W(x_{k+1}; r)}{\varepsilon}$ and, by transitivity, $x \, R \, \bar{x}$ follows. $\square$