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OPTIMAL ENVIRONMENTAL TAXES: EFFECTS OF POLLUTION DECAY AND CONSUMER AWARENESS

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ABSTRACT. The effects of nonlinear decay and consumer preferences are analyzed in a setting where optimal extraction of nonrenewable resources is combined with stock externalities. The control is exercised via a corrective tax and the time horizon is divided into two periods: an initial phase with extraction and a terminal phase without extraction. The time horizon with extraction is determined endogenously. The model does not assume separability of the objective function. The purpose here is to demonstrate that relatively simple deviations from the standard assumptions, such as linear decay and no consumer awareness, may have large effects. Sensitivity analyses indicate large differences in the optimal extraction period, the total level of extraction and cumulative emissions depending on the form of the decay function and the presence of consumers' awareness for the environment.

KEY WORDS: Global warming, fossil fuel extraction, dynamic optimization.

1. Introduction. Given the potential effects of climate change, a great deal of attention has been focused on the derivation of optimal carbon taxes (Nordhaus [1982; 1991a, b], Peck and Teisberg [1992], Sinclair [1994], Wirl [1994a, b; 1995], Rubio and Escriche [2001], Pizer [2002], van der Zwaan et al. [2002]) to correct for the stock externality associated with greenhouse gas (GHG) emissions. Some papers have explicitly linked the corrective taxes to the optimal exploitation of nonrenewable resources (Sinclair [1992], Falk and Mendelsohn [1993], Withagen [1994], Ulph and Ulph [1994], Farzin [1996], Farzin and Tahvonen [1996], Hoel and Kverndokk [1996], Tahvonen [1997]), but very few papers have evaluated the effect of nonlinear decay of GHG emissions on the optimal tax (Farzin and Tahvonen [1996], Sandal et al. [2003]). This deficiency in the existing literature is important as the uptake of

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atmospheric carbon is nonlinear in terms of cumulative carbon emissions (Joos et al. [1996]).

The main purpose of this article is to demonstrate that relatively simple deviations from the standard assumptions (linear decay and no consumer awareness) may have large effects. To address the dynamic tax problem and assess the affects on nonlinear decay on the time path of corrective taxes and cumulative emissions, an optimal feedback control law is developed and some of its important features are derived. This means that, unlike existing approaches that mitigate climate change, our approach makes it possible to determine the optimal corrective tax as a function of the level of cumulative pollution. The approach developed in this paper can be applied to maximize consumer surplus, producer surplus, or both. The application of the method to climate change is a genuine example of adaptive regulation. In each period, when new information on cumulative emissions is available (see also Sandal and Steinshamn [1998]), the corrective tax is adjusted. This approach provides insights for setting of taxes to address the potential problems of climate change. Sensitivity analysis suggests that the results are economically significant for climate change as different pollution decay functions yield quite different time horizons, total level of extraction, and levels of pollution. The discount rate too plays an important role, and the results may be very sensitive to changes in the discount rate also but emphasis is here put on the decay function.

As the objective function in the model rests on the basic supply and demand functions, and as these may be general functions of the state variable, the model is particularly suited to investigate the effects of consumers' preferences on the optimal tax. By consumers' preferences is, for example, meant that the demand for a polluting product may decrease with the aggregated level of pollution. An obvious intuitive effect of having environmentally aware consumers is that there is less need for a corrective tax. In other words, the corrective tax can be lower. As we shall see, however, this also has other and less intuitive consequences.

The article is structured as follows. The feedback model and some main properties of the optimal policy rule are derived in section 2. To show the potential importance for climate change policy, the sensitivity of the results is assessed in terms of both the decay function for stock pollution and the dependence of consumer demand on cumulative emissions. Some numerical examples are used to illustrate the analytical results.

The paper concludes with an assessment of the approach and its insights in terms of mitigating the consequences of climate change and other environmental problems associated with a stock pollutant.

2. The model. The objective is to maximize accumulated welfare, defined by the function

$$W = \int_0^T e^{-\delta t}\{U(a(t), x(t)) - D(a(t))\}\,dt + \int_T^\infty e^{-\delta(T+t)}\{\widetilde{U}(t) - D(a(t))\}\,dt, \tag{1}$$

with respect to x. The variable U is the social benefit derived from consumption of the good, x. Here, x represents extraction of fossil fuels, t is time, and δ is the discount rate. The social benefit can also be affected by the aggregate level of pollution, a. In addition, we have the direct damage of a, which is the stock externality D. Further, $\widetilde{U}$ is an alternative technology that can replace fossil fuels, for example, fuel cell technology for automobiles. There is positive extraction of the resource, $x > 0$, up to time T, and zero extraction after T. Unnecessary technicalities are avoided by assuming that we can not go back to the old technology after having switched to the new one. The switching time, T, is to be determined endogenously. To determine the optimal switching time, and investigate how it is affected by the decay, is an important aim in itself. In addition, we want to determine a rather simple way to calculate the optimal corrective tax as a feedback control law, that is to find the control variable as a function of the state variable. We will neglect the possibility of having a transition period where both types of technology are in place simultaneously. From a practical point of view this amounts to assuming that the transition period is short compared to the initial phase.

We strongly emphasize that the scope of this paper is to study how the time horizon, T, and the optimal feedback policy depend on the assumptions about nonseparability in the objective function and about the decay function.

The functions U and D may, in principle, be fairly general in a. Social utility U may, for example, represent the sum of consumers' and producers' surplus. The inclusion of a in U then describes how the level of GHG affects the demand and cost structure. For example, more pollution may increase consumers' concern for the environment and hence cause a downward shift in the demand curve for the polluting product. This case will be investigated later.

Denoting the remaining stock of fossil fuels s, equation (1) must be maximized subject to the constraints[1]

$$\begin{aligned} &\dot{s} = -x, \\ &s(t) = s_0 - \int_0^t x(u)\, du \geq 0, s_0 = s(0) > 0, \qquad (2) \\ &x(t) \geq 0, \lim_{t\to\infty} a(t) = 0, \end{aligned}$$

and

$$\dot{a} = x - f(a) \qquad (3)$$

where f is the decay function. In order to reduce the number of coefficients, $\dot{a}$ and x are measured in the same units. In other words, instead of measuring consumption of fossil fuels, x, in for example oil equivalents and then converting to GHG by some factor, it is measured directly by its content of GHG. For consistency, the stock variables, a and s, are also measured in the same units, namely as GHG. Hence, $\dot{s}, \dot{a}$, and f are measured in the same units, namely GHG units per time, whereas a and s are measured in GHG units. The condition on the aggregated level of pollution, a, in (2) ensures that we only consider policies that restore a clean environment in the long run and exclude policies that produce irreversibility.

Definition 1 (The Usual Assumptions). *The following assumptions are made if nothing else is explicitly stated:*

1. *The damage function $D(a)$ is twice continuously differentiable, nondecreasing and convex on a fixed interval $A = (0, \overline{a})$ (sufficiently large) and $D(0) = 0$.*
2. *The decay function $f(a)$ is positive and twice differentiable on A, and $f(0) = 0$. Moreover, $\lim_{a\to 0} \int_a^\alpha \frac{ds}{f(s)} \to \infty$ for any $0 < \alpha \in A$.*

Possible convex parts of f are restricted by $U_x f'' \leq D''$ on $A \times B$ where $B = (0, \overline{x})$. The marginal decay rate is limited by $\frac{f'(a)+\delta}{f(a)} \leq \frac{D'(a)}{D(a)}$ for some $\delta > 0$.

(3) *The alternative utility function, $\widetilde{U}(t) > 0$, is continuously differentiable on $(0, \infty)$ and nondecreasing.*

(4) *The current utility function $U(a, x)$ is concave and twice continuously differentiable on $A \times B$. Further[2], $U_a \leq 0, U_x > 0, U_{xx} < 0$ on $A \times B$.*

The second item puts some constraints on possible convex parts of the decay function. It is, however, sufficient that it holds on the optimal path. Further, $f(0) = 0$ means that $a = 0$ is defined as the pre-industrial level of a, which is a natural steady state. The last part of item 2 limits the relative change in natural cleaning versus the relative change in disutility associated with the stock of pollution.

A rather general decay function is useful as the decay of CO_2 through photosynthesis may be a very complex process (Joos et al. [1996]). Global warming may affect the growth of forests and phytoplankton, which again affects the CO_2 level. Increased concentrations of GHG emissions may initially increase the assimilative capacity of the environment to uptake carbon due to carbon fertilization. Further increases in GHG emissions, however, that lead to even higher GHG concentrations and higher surface temperatures may eventually lead to plant die offs that could ultimately reduce carbon uptake. The fact that there is a saturation level for how much carbon the oceans can take also calls for a nonmonotone function. Obviously the decay of carbon is a complex process that can not be well represented by linear, or even monotone, functions.

The initial stock of fossil fuel, s_0, is given. There exists an exogenously given stock level below which the costs of extracting are so high that no extraction takes place. By rescaling units, this level is defined as zero. Hence s_0 represents extractable reserves. The level $a = 0$ is defined as the natural, pre-industrial level of CO_2, which is a natural steady state and does not harm the global climate. Thus $f(0) = 0$ and $D(0) = 0$, and after extraction has terminated a will gradually approach zero.

By letting the benefit function represent the sum of producers' and consumers' surplus it can be written

$$U(a,x) = \int_0^x [P(a,y) - C(a,y)]\, dy \tag{4}$$

where P is the inverse demand function and C is the marginal cost of extraction, which is the market supply.

A more general formulation of the model would be to include s explicitly in U, but this would complicate the calculations considerably compared to the gain and blur our focus. Therefore the simplifying assumption is made that U is independent of s for $s \geq 0$ and that extraction costs increase to infinity ($U = -\infty$) for $s < 0$. Remember that s has been rescaled accordingly.

At any point in time market clearing is assumed, implying that the equilibrium level of x is given by $P = C$ without any policy measures. In other words, C is the market supply of x in a competitive economy. A competitive supply of fossil fuel is assumed throughout the paper.

In the literature it is quite common to choose objective functions that are quadratic both in the control variable and the state variable, and constraints that are linear in both (so-called linear-quadratic models) for mathematical convenience. In the present model both the objective function and the dynamic constraint are fairly general in the state variable, a. In other words, it is assumed that demand can be affected in a rather general way by the level of pollution due to changes in environmental concern among consumers among other reasons.

The externality indicates that there is need for some policy instrument in the form of quotas or corrective taxes. In this paper we use an unit tax defined by

$$\tau(a,x) = U_x(a,x) = P(a,x) - C(a,x) \quad \text{for} \quad t \leq T. \tag{5}$$

Here C is the producer price and P is the consumer price. Note that maximizing the sum of the consumers' surplus, the producers' surplus and the government's surplus, which is the tax revenue, is equivalent to maximizing $U - D$ (see Appendix 1).

It is important to keep in mind that the instrument is in effect only during the initial period with extraction. As the corrective tax is on extraction, x, it is not possible to levy any tax when $t > T$ even though the harmful effects, $D(a)$, persist into this period. An optimal tax in the initial period, therefore, also must take into account the stock externality in the terminal period. It does not matter whether τ or x is chosen as control variable in the mathematical model. The approach taken here is that the optimal extraction level, x, is found and substituted into (5) in order to find the optimal tax.

The time at which it is optimal to stop extraction is determined by the value of the alternative technology, $\widetilde{U}(t)$. The time-dependence in $\widetilde{U}$ represents technological development, and it is therefore assumed that $\widetilde{U}$ is nondecreasing.

Let $\mathcal{H} = \mathcal{H}(t, a, s, x, m, n)$ denote the current value Hamiltonian and let m and n denote the current value costate variables associated with pollution, a, and the remaining extractable resource, s, respectively. The necessary conditions are summarized in Table 1. A scrap value formulation of the problem is developed in Appendix 2. The existence of an optimal policy using the classical Filippov–Cesari existence proof and the scrap value formulation is given in Appendix 3. In this appendix it is also shown that an Arrow-type sufficient condition is satisfied.

The interpretations of the costate variables are that m is the shadow cost of pollution whereas n is the shadow price of the resource.

In addition to this, there is the requirement that the Hamiltonian and the state and costate variables are continuous at all times including T. The state variables in this maximization problem are a and s. As the stock of fossil fuel, s_0, is limited, the system will not settle on a nontrivial steady state.

Proposition 1. *The optimal tax is, in general, equal to the sum of the absolute values of the two shadow prices.*

Proof. This follows directly from the first-order conditions $\tau = U_x = n - m$, see Table 1 and the fact that $m < 0$ as proven in Lemma 1.

This is an important fact and it is often ignored in the literature (see e.g. Sinclair [1992], Ulph and Ulph [1994] and Tahvonen [1997]). The result that the tax is equivalent to the negative of the shadow price of pollution derived from simpler models can not be transferred

TABLE 1. Definitions and necessary conditions.

Description	Initial period	Terminal period
Time	$0 \leq t \leq T$	$t > T$
consumption	$x > 0, x(T) = x_T \geq 0$	$x = 0$
Tax	$\tau = n - m$	$\tau = 0$
Social welfare	$U(a,x) - D(a)$	$\widetilde{U}(t) - D(a)$
Dynamic constraint 1	$\dot{a} = x - f(a)$	$\dot{a} = -f(a)$
Dynamic constraint 2	$\dot{s} = -x$	$\dot{s} = 0$
Hamiltonian	$\mathcal{H} = U(a,x) - D(a) +$ $m \cdot [x - f(a)] - n \cdot x$	$\mathcal{H} = \widetilde{U}(t) - D(a)$ $-m \cdot f(a)$
$x = \arg\max \mathcal{H}, x \geq 0$	$m - n = -U_x, x > 0$	$m - n \leq 0, x = 0$
Costate equation 1	$\dot{m} = [\delta + f'(a)] \cdot m$ $+ D'(a) - U_a$	$\dot{m} = [\delta + f'(a)] \cdot m$ $+ D'(a)$
Costate equation 2	$\dot{n} = \delta n$	$n \cdot s = 0, n \geq 0$

to more complex models. It is, in fact, only correct in pure pollution models (with only one state variable) and with separability in pollution and consumption. If there is either another state variable that is dynamically affected by consumption or there is nonseparability in consumption and pollution, the tax is no longer equal to the shadow price of pollution. A simple example of this is if there is a flow externality in addition to the stock externality in pure pollution models. Then there must be an additional term in the tax to account for this externality (see Sandal and Steinshamn [1998], or Sandal et al. [2003]).

2.1 Matching conditions. This section concentrates on the matching that takes place at the switching time T. By defining zero as the pre-industrial level of CO_2, which is the natural steady state, we have $f(0) = 0$. We introduce two important quantities, Ψ and Ω, that are key expressions in the matching conditions at the switching time T by

$$\Psi(a;\alpha) \equiv \int_a^\alpha \frac{ds}{f(s)} \quad \text{and} \tag{6}$$
$$\Omega(a,m;\alpha) = m \cdot f(a) + e^{\delta\Psi(a;\alpha)} \int_0^a e^{-\delta\Psi(s;\alpha)} D'(s)\, ds.$$

We suppress the dependence on the constant α and use for ease of notation $\Psi = \Psi(a)$ and $\Omega = \Omega(a,m)$.

The following proposition characterizes the shadow price on pollution in the second phase:

Proposition 2. *The quantities* $t - \Psi, e^{-\delta t}\Omega$ *and* $e^{-\delta\Psi}\Omega$ *are constants along the optimal path for* $t > T$.

Totally differentiating the first two expressions with respect to time yields the result directly. The constancy of the third follows from the other two. The first follows from noticing that $\dot{\Psi} = 1$. The second results from:

$$\begin{aligned}\dot{\Omega} &= \dot{m}f + m\dot{f} + \delta\dot{\Psi}(\Omega - mf) + e^{\delta\Psi} D' \cdot e^{-\delta\Psi} \cdot \dot{a} \\ &= [(\delta + f')\, m + D']\, f + mf' \cdot \dot{a} + \delta\,(\Omega - mf) + D' \cdot \dot{a} \\ &= \delta\Omega \Rightarrow \frac{d}{dt}\left[e^{-\delta t}\Omega\right] = 0.\end{aligned}$$

Let us fix $\alpha = a(T) = a_T$ in the definition of Ψ. We then get the following corollary that will be useful later:

Corollary 1. *Assuming that* $mf \to 0$ *when* $a \to 0$ *in the second phase* $(t > T)$, *then the following relationships must hold:*

$$t = T + \Psi(a) \quad \textit{and} \quad \Omega = 0 \quad \textit{or} \tag{7}$$
$$m \cdot f(a) = -e^{\delta\Psi(a)} \int_0^a e^{-\delta\Psi(s)} D'(s)\, ds.$$

Proof. The first is an immediate consequence of the definition of a_T. Letting $a \to 0$ in the expression for Ω yields:

$$\Omega = \lim_{a\to 0}\left[e^{\delta\Psi(a)}\int_0^a e^{-\delta\Psi(s)}D'(s)\,ds\right] = \lim_{a\to 0}\frac{\displaystyle\int_0^a e^{-\delta\Psi(s)}D'(s)\,ds}{e^{-\delta\Psi(a)}}$$
$$= \lim_{a\to 0}\frac{e^{-\delta\Psi(a)}D'(a)}{e^{-\delta\Psi(a)}(-\delta\Psi')} = \delta^{-1}\lim_{a\to 0} f(a)D'(a) = 0.$$

L'Hospital's rule has been applied together with the fact that $\lim_{a\to 0}\Psi(a)\to\infty$ from the Usual Assumptions (or $t = T + \Psi \to \infty \Rightarrow a \to 0$).

Notice that in the limit of vanishing discount rate the above result implies $mf + D = 0$. This result must be interpreted carefully. In the case of zero discounting the optimality notion must be modified. A frequently used alternative is the notion of Catching-Up (CU) optimality (see e.g. page 232 in Seierstad and Sydsæther [1987]).

The following lemma can now be derived:

Lemma 1. *The shadow price m is negative for all times $0 \le t < \infty$.*

From the usual conditions and equation (7) it is evident that $m < 0$ when $T \le t < \infty$. Let us therefore assume that there exists a last point in time $t_0 < T$ such that $m = 0$. The evolution equation for the shadow price at $t = t_0$ implies $\dot m = (\delta + f')m + D' - U_a = D' - U_a > 0$. This gives that $m > 0$ immediately after, contradicting the fact that $m < 0$ to the right of $t = t_0$. By continuity of the shadow price we have established Lemma 1.

Even though the result stated in Lemma 1 is to be expected from an economic point of view, it serves the purpose of being a consistency check of our modelling approach. At the core of our approach lies the problem of determining when and how the switch will take place. The key result is given in the following proposition.

Proposition 3 (Matching Conditions). *The values of a, s, x, m, and n immediately prior to the switching time $(t = T)$ as determined by the following set of equations*

$$U(a_T, x_T) = x_T \cdot U_x(a_T, x_T) + \widetilde{U}(T) \qquad (8)$$

(9)
$$n_T = m_T + U_x(a_T, x_T)$$

(10)
$$m_T \cdot f(a_T) = -\int_0^{a_T} e^{-\delta\Psi(s;a_T)} D'(s)\,ds \geq -D(a_T)$$

where n_T *and* T *satisfy*

(11)
$$\begin{cases} n_T = 0 \quad \textit{and} \quad s_0 > \displaystyle\int_0^T x\,dt \\ n_T > 0 \quad \textit{and} \quad s_0 = \displaystyle\int_0^T x\,dt \end{cases}$$

(12)
$$a_t - a_0 = \int_0^T (x - f)\,dt.$$

The function $x = X(a)$ represents the optimal feedback solution for the consumption that, at this point, is assumed to be known. In the next section we give the appropriate boundary value problem for the optimal feedback control law. The five relations in Proposition 3 determine in principle T, a_T, x_T, m_T, and n_T when $X(a)$ is known. Because $X(a)$ will be known as a functional of these parameters it will lead to a nontrivial boundary value problem (BVP). Calculating actual values for these parameters is therefore a formidable task. This explains why much of the work in this field assumes some of these parameters exogenously given in such away that the problem is reduced to a straightforward initial value problem. In the numeric examples we will calculate all parameters for some particular cases.

Proposition 3 is derived as follows. Equation (8) follows from continuity of the Hamiltonian (see Table 1). The interpretation of this condition is that the difference in utility between the new and the old technology $(U - \widetilde{U})$ shall equal the consumption at T valued by the marginal utility, which again is equal to the optimal tax. At the switching time U is still greater than $\widetilde{U}$, but the difference compensates for the future damage of the last produced units. That is, just prior to switching, an additional polluting unit should account for the future cost of the associated pollution. Thus it is to be expected that $U - \widetilde{U} > 0$ at the switching time $t = T$.

Equation (9) follows from continuity of the costates and $\tau = U_x = n - m$. which holds throughout the first phase as it follows from the condition that the consumption should maximize the Hamiltonian. The interpretation is simply that the marginal benefits and costs must balance each other.

Equation (10) follows directly from Equation (7). The relations in (11) follow from the transversality condition on n_T and the boundary conditions on s. Equation (12) follows from $\dot{a} = X(a) - f(a)$.

The proposition also follows from standard transversality conditions for an associated finite horizon problem with a salvage value. This is shown in Appendix 2.

Corollary 2. *If at least one parameter not explicitly contained in the decay or damage function is such that a change in this parameter implies a change in the stock of pollution at the switching time, then the following holds*

$$f'(a_T) + \delta \geq 0 \Rightarrow \frac{dm_T}{da_T} \leq 0.$$

Proof. See Appendix 4.

This condition will almost always be fulfilled as a change in one of the parameters in either of the utility functions will normally imply a change in the pollution stock.

Corollary 3. *If the condition in Corollary 2 holds and the parameter is confined to $\widetilde{U}$, then $\frac{d\widetilde{U}}{da_T} \leq 0$ holds when $n = 0$. In the typical case where $U_{ax} \leq 0$, then $\frac{dx_T}{da_T} \geq 0$ also holds.*

Proof. See Appendix 4. The intuition behind this is quite clear. The better the alternative technology is, the less the aggregated stock at the switching time needs to be. The second part says that in the case of consumer awareness, consumption at the switching time will be higher the higher the aggregated level of pollution is.

Corollary 4. *At the time of transition there will be a drop in utility:* $\tau_T x_T = U(a_T, x_T) - \widetilde{U}(T) > 0$.

Proof. This follows from (8) as follows: $U(a_T, x_T) - \widetilde{U}(T) = x_T U_x(a_T, x_T) > 0$ as $x_T > 0$. If $x_T = 0$ then $\widetilde{U}(T) = U(a_T, 0) = 0$, but $\widetilde{U} > 0$ by definition.

This is a quite surprising result as intuitively one would perhaps expect continuity in utility along an optimal path. It follows, however, from continuity of the Hamiltonian.

2.2 Concavity of the maximized Hamiltonian. The Hamiltonian in the first phase, $0 \leq t \leq T$, is also the Hamiltonian for the alternative formulation with scrap value. Both formulations yield the same necessary conditions. Assume at this point that the solution to the necessary conditions has been found. It will be demonstrated that such a solution is optimal as it satisfies an Arrow-type sufficiency theorem. In order to do so, we must show that the maximized Hamiltonian is concave in the state space under consideration. Other details are given in Appendix 3. The current value Hamiltonian is given in Table 1. There is an interior unique solution to $H_x = 0$ as $U_x > 0$:

$$H_x = U_x(a, x) + m - n = 0 \Rightarrow x = X(a, m - n) \quad \text{and} \tag{13}$$

$$X_a = -\frac{U_{ax}}{U_{xx}}.$$

The maximized Hamiltonian is given by

$$H^0(a, m, n) = H(a, X(a, m - n), m, n).$$

Differentiating with respect to a and using (13):

$$\begin{aligned} H_a^0 &= H_a(a, X, m, n) + H_x(a, X, m, n) X_a(a, m - n) = H_a(a, X, m, n) \\ H_{aa}^0 &= H_{aa}(a, X, m, n) + H_{ax}(a, X, m, n) X_a \\ &= U_{aa}(a, X) - D''(a) - m f''(a) + U_{ax}(a, X) X_a, \end{aligned}$$

or in more suitable form:

$$H_{aa}^0 = \frac{U_{aa}U_{xx} - U_{ax}^2}{U_{xx}} + [-D'' - mf''] .$$

It is now straightforward to prove that

$$H_{aa}^0 \leq \frac{U_{aa}U_{xx} - U_{ax}^2}{U_{xx}} \tag{14}$$

by using the fact that $m < 0$. This is seen by the following reasoning:

(1) In any region where $f'' \leq 0$ it is trivially true as both terms in the square brackets are negative because $D'' \geq 0$ by the Usual Assumptions.
(2) In the rest of the state space therefore $0 \leq -mf'' = (U_x - n)f'' \leq U_x f'' \leq D''$ or $-mf'' - D'' \leq 0$. The optimal condition $m - n = -U_x$ from (13) has been used in the equality, and the nonnegativity of the shadow price has been used in the next inequality. The last inequality stems from Usual Assumption 2 that restricts the strength of the convexity of f.

The right-hand side of (14) is nonpositive due to the concavity of U.

2.3 The optimal path. In order to study the optimal path of the control variable x, and the corresponding tax, it is useful to derive the optimal control as a function of the state variable, that is, as a feedback control law. A feedback control law represents an adaptive regulation as the optimal tax is directly affected by changes in the environment. The tax level is determined as soon as the level of pollution is estimated.

Proposition 4. *The condition* $0 > \frac{dX}{da} > -\frac{U_{ax}}{U_{xx}}$ *is necessary and sufficient for* $\dot{\tau} \cdot \dot{x} > 0$ *when* $\dot{a} \neq 0$, *in other words, for the same kind of monotonicity both in the tax and the consumption during an interval. Further, the tax will be nonincreasing wherever consumption has a possible critical point, and consumption will be nonincreasing wherever the tax has a possible critical point.*

Proof. By definition $\dot{\tau} = \dot{U}_x(a, x) = \dot{a}U_{xx}(\frac{dX}{da} + \frac{U_{ax}}{U_{xx}})$, and hence

$$\dot{\tau} \cdot \dot{x} = \dot{a}^2 U_{xx} \left(\frac{dX}{da} + \frac{U_{ax}}{U_{xx}} \right) \frac{dX}{da} > 0.$$

From this it is seen that as $U_{ax} = P_a - C_a \leq 0$ and $U_{xx} < 0$, then we can not have $\frac{dX}{da} \geq 0$. Further, the parenthesis can not be negative. This establishes the proposition.

This proposition is quite interesting because it is counterintuitive, and it can never happen without explicit a-dependence in the utility. In most of the literature there is no such explicit a-dependence as the welfare usually is separable in consumption and pollution. Intuitively one would expect that consumption goes down when there is put a tax on it. Here this does not necessarily happen. The tax and consumption may go the same way for a while. The reason is that with a-dependence in utility, for example due to consumer awareness, consumption may go down simultaneously with a decrease in the tax as long as the pollution still is increasing. This is because consumption is more directly affected by the pollution level. This is not only a possibility but, as shown by the numerical examples, it may also very well happen in reality.

The differential equation governing the optimal feedback rule for the control variable, x, is readily derived. Both from a mathematical and economic perspective it is useful to define the following scalar functions:

$$\begin{aligned} &S(a) \equiv U(a, f(a) - D(a), \\ (15) \qquad &\mathcal{L}(a,x) \equiv U(a,x) - U_x(a,x) \cdot (x - f(a)) - U(a, f(a)), \\ &\mathcal{P}(a,x) \equiv \mathcal{L}(a,x) + S(a). \end{aligned}$$

The economic interpretation of S is that it represents the level of social benefit that can be obtained at any time by fixing the level of pollution by producing $x = f$ such that $\dot{a} = 0$. $\mathcal{L}(a,x)$ holds a potential utility gain much the same way that a moving physical object holds a potential of doing work (its energy content) strictly associated with its movement. Its value is associated with the change in the pollution-level and, as such, can be viewed as a dynamic potential utility gain for changing the pollution state in the total asset.

Lemma 2. $\mathcal{L}(a,x)$ *is semi-definite and is bounded by*

$$\underline{M}(a)(x-f)^2 \leq \mathcal{L}(a,x) \leq M(a)(x-f)^2$$

where

$$2\underline{M} = \min |U_{xx}(a,x)| \tag{16}$$

$$2M = \max |U_{xx}(a,x)| \tag{17}$$

on $x \in [0, \overline{x}]$.

The semi-definite property is a direct result of The Usual Assumptions, in particular the regularity and the concavity of U with respect to x. Taylor expansion yields $U(a,f) = U(a,x) + U_x(a,x) \cdot (f-x) + \frac{1}{2}U_{xx}(a,s) \cdot (f-x)^2$ for $f \leq s \leq x$ and hence $\mathcal{L}(a,x) = -\frac{1}{2}U_{xx}(a,s) \cdot (f-x)^2$.

Lemma 3. *$\mathcal{P}$ is equal in value (but not as a function) with the quantity $\mathcal{H} + nf$, and it satisfies the relation $\dot{\mathcal{P}} = [nf' - \delta U_x]\,\dot{a}$ on any part of an optimal path in the first phase.*

The first result follows from noticing that $\mathcal{H} + nf = U(a,x) - D(a) + (m-n) \cdot [x - f(a)]$. Inserting the first order condition for $m - n$ from Table 1 results in $\mathcal{H} + nf = U(a,x) - D(a) - U_x \cdot [x - f(a)] = \mathcal{L} + S$. The last part follows from differentiating the above result and applying the first order conditions:

$$\begin{aligned}
\dot{\mathcal{P}} &= \dot{\mathcal{H}} + \dot{n}f + n\dot{f} = \mathcal{H}_a\dot{a} + \mathcal{H}_x\dot{x} + \mathcal{H}_m\dot{m} + \mathcal{H}_n\dot{n} + \dot{n}f + n\dot{f} \\
&= \mathcal{H}_a\mathcal{H}_m + \mathcal{H}_m(\delta m - \mathcal{H}_a) + \mathcal{H}_n\delta n + \delta nf + nf'\dot{a} \\
&= \mathcal{H}_m\delta m - (x-f)\delta n + nf'\dot{a} = \delta\,(m-n)\,\dot{a} + nf'\dot{a} \\
&= [-\delta U_x + nf']\,\dot{a}.
\end{aligned}$$

The quantity $\mathcal{P}$ can be interpreted as the total rent less the resource rent. This lemma gives us an important tool for producing the optimal feedback policy.

The problem initiated in this paper can now be stated as the following boundary value problem:

Proposition 5 (Boundary value problem). *The first-order condition for the control problem defined through equations (1), (2), and (3) implies the boundary value problem given by*

$$
\begin{aligned}
\dot{a} &= x - f \\
-U_{xx}\dot{x} &= -\mathcal{P}_a - \delta U_x + nf' \qquad (18)\\
\dot{n} &= \delta n,
\end{aligned}
$$

and the boundary conditions are given by the relations stated in the Matching Conditions together with the initial condition on a.

Proof. We only need to show the differential equation for the consumption, x. The other two are already given in Table 1. Differentiating the equation stating that we have an inner optimum for $t < T$, and using the other first-order conditions (see Table 1), implies

$$
\begin{aligned}
\frac{d}{dt}U_x = \frac{d}{dt}(n-m) &= \dot{n} - \dot{m} \\
&= \delta n - [(\delta + f')m - U_a + D'] \\
&= \delta(n-m) - f'm + U_a - D' \\
&= (\delta + f')(n-m) - nf' + U_a - D' \\
&= (\delta + f')U_x + U_a - D' - nf' \\
&= \delta U_x - nf' - D' + U_a + f'U_x \\
&= \delta U_x - nf' + [\mathcal{P}_a + (x-f)U_{xa}] .
\end{aligned}
$$

Finally, inserting $\frac{d}{dt}U_x = U_{xx}\dot{x} + U_{xa}\dot{a} = U_{xx}\dot{x} + U_{xa}(x-f)$ completes the derivation. Notice that Lemma 3 yields

$$
\begin{aligned}
(19) \qquad \dot{\mathcal{P}} = \mathcal{P}_a\dot{a} + \mathcal{P}_x\dot{x} = \mathcal{P}_a\dot{a} + \mathcal{L}_x\dot{x} = [-\delta U_x + nf']\,\dot{a} \\
\Leftrightarrow \mathcal{L}_x\dot{x} = [-\mathcal{P}_a - \delta U_x + nf']\,\dot{a}.
\end{aligned}
$$

From this, we can derive

Corollary (BVP in simplified feedback form). *A direct way to obtain the optimal feedback BVP is given by the matching conditions and*

$$(20)\quad -(x-f)U_{xx}\frac{dx}{da} = -\mathcal{P}_a - \delta U_x + nf' \quad \textit{and} \quad (x-f)\frac{dn}{da} = \delta n.$$

Proof. Dividing (19) by $\dot{a}$, we get $\mathcal{L}_x = -(x-f)U_{xx} = -U_{xx}\dot{a}$, provided $\dot{a} \neq 0$. This establishes (20).

Note that all terms in (20) are expressions in either x or a, and hence this shows that our control can be expressed in a simplified feedback form without explicit reference to time.

The next proposition supply bounds on the optimal policy or the rate of change in pollution. This proposition covers the typical case where the level of pollution is nondecreasing in the period with the old polluting technology.

Proposition 6 (Bounds on x and $\mathcal{P}(a,x)$.) *The optimal policy x satisfies the following relations:*

$$(21)\quad \underline{M}(a)\left[x - f(a)\right]^2 + S(a) \leq \mathcal{P}(a,x) \leq M(a)\left[x - f(a)\right]^2 + S(a).$$

In the typical cases where $\dot{a} = x - f(a) \geq 0$ and $\delta > 0$ for $t \leq T$ the optimal policy is bounded by

$$(22)\qquad x \geq f(a) + \sqrt{\frac{\widetilde{U}(T) - \widetilde{D}(a_T) + \delta\eta(a) + nf(a) - S(a)}{M(a)}}.$$

If $f'(a) \geq 0$ the policy is also bounded by

$$(23)\qquad x \leq f(a) + \sqrt{\frac{\widetilde{U}(T) - \widetilde{D}(a_T) + \delta\xi(a) + nf(a_T) - S(a)}{\underline{M}(a)}}$$

where $\underline{M}$ and M are defined by (16) and (17) respectively, and

$$(24)\qquad \begin{aligned} \tilde{D}(a_T) &= D(a_T) - \int_0^{a_T} e^{-\delta\Psi(a;a_T)} D'(a)\,da \geq 0 \\ \eta(a) &= \int_0^{a_T} U_x(a,\overline{x})\,dz \\ \xi(a) &= \int_0^{a_T} U_x(a,0)\,da. \end{aligned}$$

Proof. Lemma 2 yields (21) directly given the definitions in (15). Lemma 3 implies

$$\begin{aligned}\mathcal{P}(a,x) &= \mathcal{P}(a_T,x_T) - [nf(a)]_T^t - \delta\int_T^t [nf + U_x(a,x)\dot{a}]\,dt \\ &\geq \widetilde{U}(T) - \widetilde{D}(a_T) + nf(a) + \delta\int_a^{a_T} U_x(a,x)\,da \\ &\geq \widetilde{U}(T) - \widetilde{D}(a_T) + nf(a) + \delta\eta(a).\end{aligned}$$

We also have when $f'(a) \geq 0$

$$\begin{aligned}\mathcal{P}(a,x) &= \mathcal{P}(a_T,x_T) - \int_a^{a_T} nf'(a)\,da + \delta\int_a^{a_T} U_x(a,x)\,da \\ &\leq \mathcal{P}(a_T,x_T) + \delta\xi(a) = \widetilde{U}(T) - \widetilde{D}(a_T) + n_T f(a_T) + \delta\xi(a).\end{aligned}$$

These inequalities together with (21) gives (22) and (23).

Note that if U is a quadratic function in x, we have $M(a) = \underline{M}(a) = -\frac{1}{2}U_{xx}(a,x)$ and $\mathcal{P}(a,x) = M(a)\,[x - f(a)]^2 + S(a)$. If, in addition, the the discount rate is 0, we have the following explicit relationship for the feedback control law:

Corollary 6. *In the limit $\delta \to 0$ and $U_{xxx} = 0$ the optimal consumption policy is given by*

$$x(a) = f(a) + \sqrt{\frac{\widetilde{U}(T) + n\cdot f(a) - S(a)}{M(a)}}.$$

Proof. Lemma 3 implies that $\mathcal{P}(a,x) - nf(a)$ is constant; that is,

$$M(a)(x-f)^2 + S(a) - nf(a) = \mathcal{P}(a_T,x_T) - n_T f(a_T),$$

which gives the result. It is worth pointing out that the upper and lower bounds in Proposition 6 are equal in this case but now it holds with no assumption about nondecreasing decay. The relevance of this is that the narrower the bounds are, the better it is, because then we know where the optimal path is to be found. When the bounds collapse, we have an exact solution for the optimal path.

TABLE 2. Economic parameters (normalized)

Parameter	Value	Parameter	Value
$p_0(a)$	16-0.00112a	p_1	0.64
c_0	1	c_1	0.05
$D(a)$	$9{\cdot}10^{-5} \cdot a^2$	$\widetilde{U}(t)$	$101 + 0.1 \cdot t$

3. Numerical examples. The purpose of this section is to illustrate some of the theoretical results by a couple of numerical examples. We do not claim that these examples represent real world situations, but we use quasi-realistic data in order for the examples to have some relevance. A quadratic damage function (stock externality) is assumed, and the effects of two different decay functions and two discount rates are investigated.

3.1 Numerical specification and results. The present level of CO_2 is assumed to be 625 Gt (Giga-tonnes) higher than the preindustrial level. Rescaling such that the preindustrial level, by definition, is zero implies $a(0) = a_0 = 625$. The remaining stock of fossil fuels is estimated to around 7,000, that is $s_0 = 7{,}000$.

The economic parameters given in Table 2 are based on short-term supply, and demand elasticities for fossil fuel are equal to 2 and -0.15, respectively, when $a = a_0$ (Burniaux et al. [1992]). As the model is adaptive, it is the short-term elasticities that are relevant. The inverse demand for fossil fuel is assumed to be linear:

$$P(a, x) = p_0(a) - p_1 \cdot x$$

where p_0 and p_1 are parameters. The marginal cost function is also assumed to be linear:

$$C(x) = c_0 + c_1 \cdot x,$$

implying that U is quadratic in x. All parameters in the demand and supply function can be made dependent on the state variable, a, if that is relevant. We have chosen only to let p_0 be a-dependent in order to concentrate on consumers' awareness through a shift in demand as a increases.

The values of the economic parameters are given in Table 2.

As seen from Table 2 the marginal cost of extraction at zero consumption has been normalized to one. The stock externality is quadratic in a. The size of this externality is uncertain, but some studies indicate that it will be around 2% of the world's gross domestic product if current emissions continue (Schelling [1997]).

Two decay functions have been used, namely the linear one, $f(a) = 0.01872 \cdot a$, and the nonlinear

$$f(a) = \begin{cases} 0.01872 \cdot a & a < 640 \\ 11.9808 & otherwise \end{cases}.$$

The nonlinear (or piecewise linear) is based on the assumption that there is a saturation level with respect to decay. Pollution decay increases at a constant rate up to a certain level, but beyond that level the decay is constant. This level has been chosen a bit higher than the present level. To assume linear decay, no matter how much we pollute, is too optimistic.

The key results with two different discount rates are reported in Table 3. The tax has been converted to an ad valorem tax as this is easier to relate to than the unit tax.

TABLE 3. Key results

	2% discounting		3% discounting	
	Linear decay	Nonlinear decay	Linear decay	Nonlinear decay
$x(0)$	15.6	14.6	16.8	15.7
$x(T)$	16.2	15.7	19.6	17.7
$a(T)$	712.7	690.4	809.6	769.3
T	26.1	22.8	45.8	33.8
$\tau_{\min}$	167%	192%	28%	104%
$\tau_{\max}$	187%	256%	146%	201%

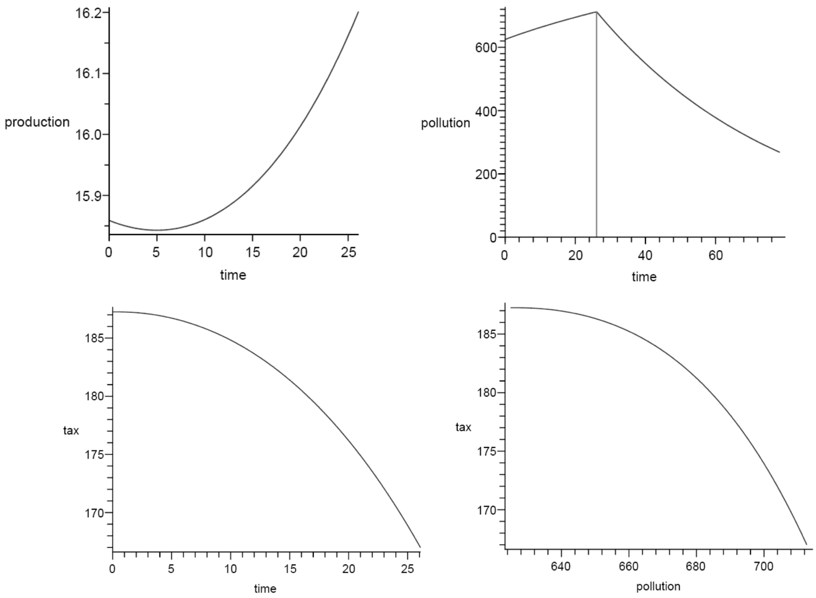

FIGURE 1. Linear decay, 2% discounting.

The results with 2% discounting, and linear and nonlinear decay respectively, are illustrated in Figures 1–2. Each figure shows pollution, consumption and the ad valorem tax as a function of time and the optimal feedback control law for the tax.

Proposition 4 is clearly illustrated in Figure 1, where consumption has a minimum at time 5, whereas, the tax is decreasing over the whole period. The second part of Proposition 4 is seen from Figure 2. Here, it is also seen how sensitive the state and control variables, as well as the switching time, are to the decay function. A less optimistic view on decay calls for a higher tax and lower consumption. Note also that tax and consumption are nonmonotone with an interior maximum and minimum respectively and, as foreseen by Proposition 4, the minimum in consumption occurs prior to the maximum in the tax. This is seen from Figure 2.

The approach has been to choose the simplest nonlinearity possible, namely piecewise linear. Even so, we get untraditional results such as a tax that is first increasing and then decreasing as seen from Figure 2.

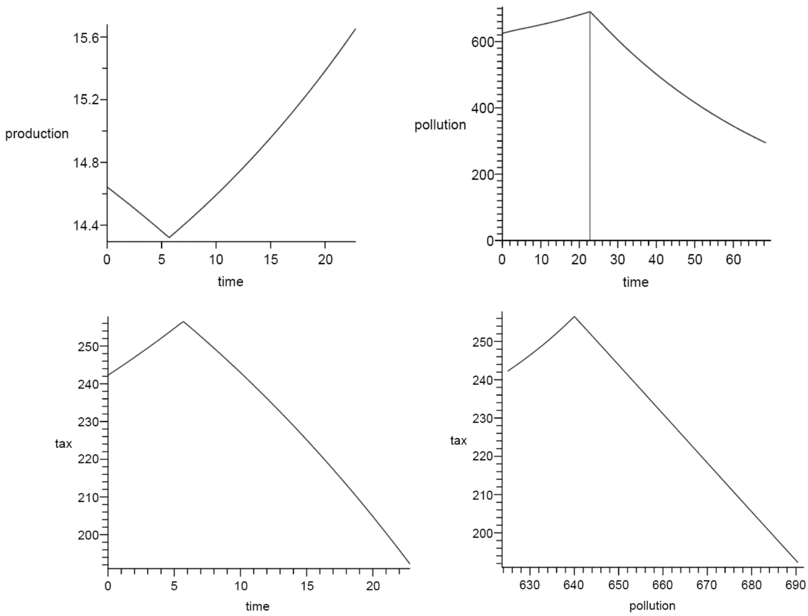

FIGURE 2. Nonlinear decay, 2% discounting.

The typical result in the literature is that the optimal tax shall decrease (e.g., Sinclair [1992, 1994]). Note, however, that at the switching time the tax is typically decreasing.

These examples are not meant to give a precise description of reality but to illustrate some of the results found in the theoretical analysis and to show that the optimal tax is sensitive both to decay and to discounting. It emphasizes the need for more empirical and theoretical studies on the decay function.

4. Conclusions. In this paper a feedback control law that can be used to control the consumption of fossil fuel products in the presence of stock externalities associated with emissions of greenhouse gases has been used to analyze the effects of nonlinear decay and discounting upon optimal carbon taxes in the case where the utility is directly affected by pollution in addition to a stock externality. The main result is that the tax is quite sensitive to both decay and discounting.

The total time horizon in the optimization problem is divided into two periods, one with extraction and emissions and one without extraction. The model has been used to analyze the time path of the corrective tax in the period with extraction by taking into account the stock externality both in this period and in the remaining period.

Special emphasis has been put on the effects of nonmonotone decay of carbon in the atmosphere and the interaction with consumer preferences. If decay is in fact monotone, the optimal period of extraction will be longer than if it is nonmonotone. The sensitivity to the shape of the decay function also stresses the importance of estimating the decay function. Assuming a linear decay function for mathematical convenience, as is often done, may represent a serious mistake if the actual decay is nonmonotone. The theoretical results are confirmed by some numerical examples.

The results in this paper confirms the results by Ulph and Ulph [1994] that the time path of the optimal path depends on the decay function and that papers claiming that the tax should decline over time, for example, Sinclair [1992], are not correct in general. However, as Ulph and Ulph look at linear decay, they conclude that positive decay makes the tax rise over time. In this paper, it is shown that the optimal tax can increase or decrease over time depending on the shape of the decay function. The numerical results are somewhat different from what Ulph and Ulph found, especially as they assume an exogenous switching time.

Appendix 1: The Objective Function

In this appendix, it is shown that the sum of consumers' surplus, producers' surplus, and the government's surplus is equal to $U - D$. If these surpluses are called CS, PS, and GS, respectively, we have by definition

$$CS \equiv \int_0^x P(a,s)\,ds - (C(a,x)+\tau)x,$$

$$PS \equiv C(a,x)x - \int_0^x C(a,s)\,ds,$$

$$GS \equiv \tau x - D(a).$$

Summing these surpluses yields

$$\int_0^x [P(a,s) - C(a,s)]\, ds - D(a).$$

Appendix 2: A Scrap Value Formulation

The purpose of this appendix is to show how the matching conditions (8)–(12) are derived using a salvage (or scrap) value approach. The present value of the last phase of the infinite time problem can alternatively be defined as the salvage value for a problem with a finite time horizon T where T is to be determined as part of the optimization.

In (6), the function $\Psi(a;\alpha)$ was defined. By setting $\alpha = a_T$, and applying Proposition 3, it is readily seen that

$$t = T + \Psi = T + \Psi(a, a_T) \text{ where } \frac{\partial \Psi}{\partial a} = -\frac{1}{f(a)} \text{ and } \frac{\partial \Psi}{\partial a_T} = \frac{1}{f(a_T)}.$$

Calling the present value of the last period φ, it can be defined and written as:

$$\begin{aligned}\varphi(T, a_T) &= \int_T^\infty e^{-\delta t}[\widetilde{U}(t) - D(a)]\, dt \\ &= \int_T^\infty e^{-\delta t}\widetilde{U}(t)\, dt + \int_0^{a_T} e^{-\delta(T+\Psi)} \frac{D(a)}{f(a)}\, da.\end{aligned}$$

Partial integration of the last term yields

$$\varphi(T, a_T) = \int_T^\infty e^{-\delta t}\widetilde{U}(t)\, dt - \frac{e^{-\delta T}}{\delta}\left[D(a_T) - \int_0^{a_T} e^{-\delta\Psi} D'(a)\, da\right].$$

The transversality conditions involve $\frac{\partial\varphi}{\partial T}$ and $\frac{\partial\varphi}{\partial a_T}$. Straightforward calculations yield

$$\begin{aligned}\frac{\partial \varphi}{\partial T} &= -e^{-\delta T}\left[\widetilde{U}(T) - D(a_T) + \int_0^{a_T} e^{-\delta\Psi} D'(a)\, da\right] \\ \frac{\partial \varphi}{\partial a_T} &= -\frac{e^{-\delta T}}{f(a_T)} \int_0^{a_T} e^{-\delta\Psi} D'(a)\, da.\end{aligned} \tag{25}$$

Notice that $\dot{\varphi}(T, a_T) = -e^{-\delta T}[\widetilde{U}(T) - D(a_T)]$ from its definition. The Hamiltonian in this case is given by

$$H(t, a, s, x, m, n) = e^{-\delta t}\,[U(a, x) - D(a)] + m \cdot (x - f(a)) - n \cdot x.$$

The transversality condition on the shadow price of pollution is $m_T = \frac{\partial \varphi}{\partial a_T}$ or

$$e^{\delta T} m_T\, f(a_T) = -\int_0^{a_T} e^{-\delta \Psi} D'(a)\, da, \tag{26}$$

which is recognized as matching condition (10). The transversality condition associated with a free time horizon is $H + \frac{\partial \varphi}{\partial T} = 0$ at $t = T$. Before we apply this relationship, we use the fact that the optimal policy is, by definition, an interior solution which implies:

$$m - n = -e^{-\delta t} U_x(a, x). \tag{27}$$

Inserted into the Hamiltonian this yields $H = e^{-\delta t}\,[U - D - xU_x] - m \cdot f$. At the end of the first period, $t = T$, we get from (26):

$$\begin{aligned} H + \frac{\partial \varphi}{\partial T} &= e^{-\delta T}\left[U - D - xU_x + \int_0^{a_T} e^{-\delta \Psi} D'(a)\, da\right] \\ &\quad -e^{-\delta T}\left[\widetilde{U} - D + \int_0^{a_T} e^{-\delta \Psi} D'(a)\, da\right] \\ &= e^{-\delta T}[U - xU_x - \widetilde{U}]. \end{aligned}$$

The transversality condition on the Hamiltonian therefore implies that

$$U(a_T, x_T) - x_T U_x(a_T, x_T) - \widetilde{U}(T) = 0,$$

which is matching condition (8) in the main text.

Continuity of the shadow values at the end point $t = T$ applied to (27) yields matching condition (9). Condition (11) is a direct consequence of the transversality condition on the costate variable n, and condition (12) follows directly from the dynamic equation for the pollution level (3).

Appendix 3: An Existence Proof and an Arrow-Type Sufficiency Result

In this section, we show that our problem has a solution. We apply the Filippov–Cesari existence theorem as it is given in theorem 6.18 in Seierstad and Sydsæther [1987]. The time interval of interest is taken to be $[0, \widehat{T}]$ for sufficiently large $\widehat{T}$. All the conditions in the theorem are trivially satisfied except, possibly, for the convexity of the set $N(a, t, x)$ and the assumed upper constant bound on the state variables. The latter is straightforward as can be seen from the fact that $\dot{a} + \dot{s} = -f(a)$ implies $0 \leq a + s \leq a_0 + s_0$. The theorem assumes that the set

$$N(a, t, x) = \{(e^{-\delta t}[U(a, x) - D(a)] + \gamma, x - f(a), -x) \\ : \gamma \leq 0, x \in [0, \hat{x}]\}$$

is convex for all $(a, t) \in R \times [0, \widehat{T}]$. We fix (a, t) and let $y_i = e^{-\delta t}[U(a, x_i) - D(a)] + \gamma_i$ and $\gamma_i \leq 0$ for $i = 1, 2$ and let x_3 and y_3 be the convex combinations of x_1, x_2 and y_1, y_2. From the concavity of U, we have that the convex combination $\lambda(y_1, x_1 - f(a), -x_1) + (1 - \lambda)(y_2, x_2 - f(a), -x_2) = (\lambda y_1 + (1 - \lambda)y_2, x_3 - f(a), -x_3) = (y_3, x_3 - f(a), -x_3)$ and $y_3 = e^{-\delta t}[\lambda U(a, x_1) + (1 - \lambda)U(a, x) - D(a)] + \lambda\gamma_1 + (1 - \lambda)\gamma_2 \leq e^{-\delta t}[U(a, x_3) - D(a)] + \lambda\gamma_1 + (1 - \lambda)\gamma_2$, implying that $\gamma_3 \leq \lambda\gamma_1 + (1 - \lambda)\gamma_2 \leq 0$ and thereby $(y_3, x_3 - f(a), -x_3) \in N(a, t, x)$. Hence the set is convex.

The Arrow-type sufficiency result is based on note 6.20 in Seierstad and Sydsæther [1987]. This note deals with a more general problem than the present one. In addition to the concavity of the maximized Hamiltonian, we need to show that the scrap value is concave with respect to the state variables. All other conditions are trivially satisfied. The model presented in this paper has a simpler structure than the setting in the referenced note. We deal with simple state constraints in the form of nonnegativity conditions and no combined constraints on state and policy.

The concavity of the scrap value $\varphi(T, a_T)$ is shown by differentiating (25). We obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial a_T^2}\varphi(T, a_T)\\
&\quad = e^{-\delta T}\left[\frac{f'(a_T)+\delta}{f(a_T)^2}\int_0^{a_T} e^{-\delta\Psi(a;a_T)}D'(a)\,da - \frac{D'(a_T)}{f(a_T)}\right]\\
&\quad = \frac{e^{-\delta T}}{f(a_T)}\int_0^{a_T} e^{-\delta\Psi(a;a_T)}D'(a)\,da\cdot\left[\frac{f'(a_T)+\delta}{f(a_T)} - \frac{D'(a_T)}{D(a_T)}\right]\le 0.
\end{aligned}$$

The last term in the last square brackets stems from noticing that

$$\int_0^{a_T} e^{-\delta\Psi(a;a_T)}D'(a)\,da \le D(a_T)$$

and the inequality is a direct consequence of the constraining relation on the marginal decay function in The Usual Assumptions.

Appendix 4

Proof of Corollary 2. Differentiating the first part of (10) w.r.t a_T and using the inequality part yields:

$$\begin{aligned}
&\frac{dm_T}{da_T}\cdot f(a_T) + m_T\cdot f'(a_T)\\
&\qquad\qquad = -D'(a_T) + \frac{\delta}{f(a_T)}\int_0^{a_T} e^{-\Psi(s;a_T)}D'(s)\,ds\\
&\qquad\qquad = -D'(a_T) - \delta m_T\\
&\frac{dm_T}{da_T}\cdot f(a_T) = -(\delta + f'(a_T))m_T - D'(a_T)\\
&\qquad\qquad \le \frac{f'(a_T)+\delta}{f(a_T)}D(a_T) - D'(a_T)\\
&\qquad\qquad = D(a_T)\left[\frac{f'(a_T)+\delta}{f(a_T)} - \frac{D'(a_T)}{D(a_T)}\right]\le D(a_T)\cdot 0 = 0.
\end{aligned}$$

The last inequality follows from part 2 of Definition 1.

Proof of Corollary 3. As $n \equiv 0$ we have $m_T = -U_x(a_T, x_T)$. This inserted into (8) yields (at time T) $U(a,x) = -x \cdot m(a) + \widetilde{U} \Rightarrow U_a\,da + U_x\,dx = -m\,dx - x\,dm + d\widetilde{U}$ or $[U_a + \frac{dm}{da}]\,da = d\widetilde{U}$. As both terms in square brackets are nonpositive, we have the first result. The second result (also at time T) follows from $m(a) = -U_x$ yielding $(-\frac{dm}{da})\,da = (-U_{xx})\,dx + U_{ax}\,da$, which can be rewritten $[-\frac{dm}{da} - U_{ax}]\,da = (-U_{xx})\,dx$. Consumer awareness is equivalent to $U_a < 0$ and hence $\frac{d\widetilde{U}(T)}{da_T} > 0$.

ENDNOTES

1. Dots denote time derivatives.
2. Subscript denotes partial derivative

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